Argumentation and mathematical proof

Alison Pease

Argumentation Research Group

University of Dundee

In collaboration with Ursula Martin (U of Oxford), Andrew Aberdein (Florida - FIT), Simon Colton (Goldsmiths), Alan Smaill (U of Edinburgh) and John Lee (U of Edinburgh)

My background

- Mathematics/Philosophy
- Post-compulsory education (16-19) PGCE
- MSc in AI: *An Investigation into Philosophical Dialectic*
- PhD in AI: *A Computational Model of Lakatos-style Reasoning*

My research

Cognitive aspects of mathematical reasoning (DReaM Group)

Computational Creativity Project (Computational Creativity Research Group)

Crowdsourced Math Project (Theory Research Group)

Argumentation and mathematics (Arg-Tech)

To come...

- **Part I:** the relationship between argumentation and mathematical proof
- **Part II:** philosophical and linguistic aspects of the process of constructing and presenting mathematical proof
- **Part III:** hands-on analysis of mathematical and non-mathematical arguments

Part I: the relationship between argumentation and mathematical proof

Deduction is the only necessary reasoning. It is the reasoning of mathematics. It starts from a hypothesis, the truth or falsity of which has nothing to do with the reasoning; and of course its conclusions are equally ideal. [. . .]

Peirce, C. S. (1931–58). *Collected Papers of Charles Sanders Peirce*. Harvard University Press, Cambridge, Mass. Eight Volumes [58, 5.145].

Mathematical arguments alone seem entirely safe: given the assurance that every sequence of six or more integers between 1 and 100 contains at least one prime number, and also the information that none of the numbers from 62 up to 66 is a prime, I can thankfully conclude that the number 67 is a prime; and that is an argument whose validity neither time nor the flux of change can call in question. This unique character of mathematical arguments is significant. Pure mathematics is possibly the only intellectual activity whose problems and solutions are 'above time'. A mathematical problem is not a quandary; its solution has no time-limit; it involves no steps of substance. As a model argument for formal logicians to analyse, it may be seducingly elegant, but it could hardly be less representative.

S. Toulmin. *The uses of argument*. CUP, Cambridge, (1958).

Corollary 3.3. *Let ν and $\tilde{\nu}$ be the equal-slices and non-degenerate equal-slices measures on $[k]^n$, respectively. Then for any set $A \subset [k]^n$ we have $|\nu(A) - \tilde{\nu}(A)| \leq k^2/n$*

Proof. It follows from Lemma 3.2 that the probability that a slice is degenerate is at most k^2/n. Therefore, if A is a set that consists only of non-degenerate sequences, then its non-degenerate equal-slices measure is $(1-c)^{-1}$ times its equal-slices measure, for some $c < k^2/n$. Therefore, for such a set, $0 \leq \tilde{\nu}(A) - \nu(A) = c\tilde{\nu}(A) \leq k^2/n$. If A consists only of degenerate sequences, then $0 \leq \nu(A) - \tilde{\nu}(A) = \nu(A) \leq k^2/n$. The result follows, since if one takes a union of sets of the two different kinds, then the differences cancel out rather than reinforcing each other. □

For later use, we slightly generalize Lemma 3.2.

Lemma 3.4. *Let x be chosen randomly from $[k]^n$ using the equal-slices distribution. Then the probability that fewer than m coordinates of x are equal to k is at most mk/n.*

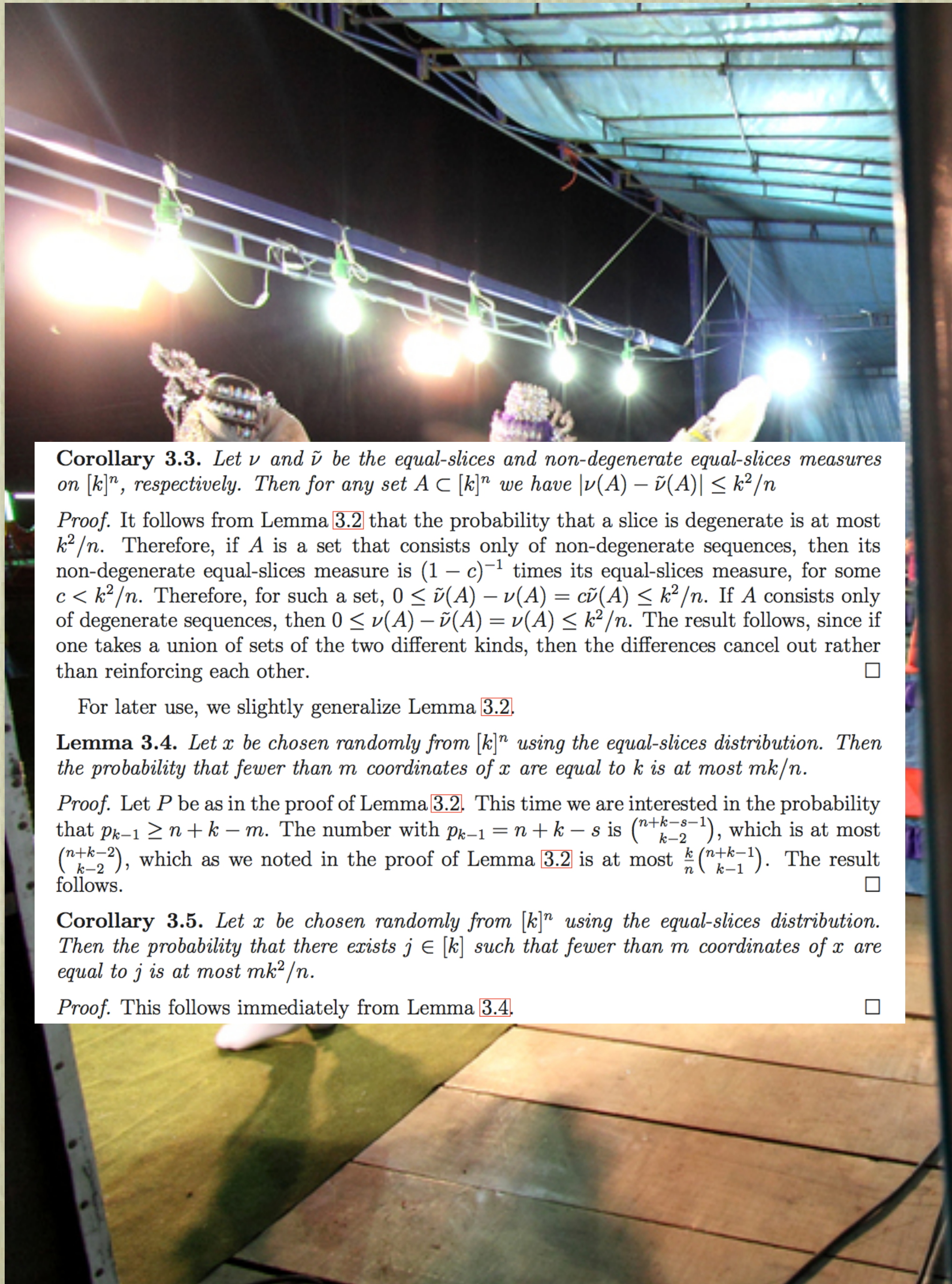
Proof. Let P be as in the proof of Lemma 3.2. This time we are interested in the probability that $p_{k-1} \geq n+k-m$. The number with $p_{k-1} = n+k-s$ is $\binom{n+k-s-1}{k-2}$, which is at most $\binom{n+k-2}{k-2}$, which as we noted in the proof of Lemma 3.2 is at most $\frac{k}{n}\binom{n+k-1}{k-1}$. The result follows. □

Corollary 3.5. *Let x be chosen randomly from $[k]^n$ using the equal-slices distribution. Then the probability that there exists $j \in [k]$ such that fewer than m coordinates of x are equal to j is at most mk^2/n.*

Proof. This follows immediately from Lemma 3.4. □

Specifically, not all—indeed hardly any—mathematical proofs are strict formally valid logical derivations. Of course, most of them can be restated in this manner, sometimes with comparatively little effort, but this is not something that mathematicians routinely do. To insist on such paraphrase is to misrepresent the nature of mathematical practice. Moreover, there is much that mathematicians do besides proving results, central as that activity may be. Most of this work may still be understood, however, as a species of *argument*.

Mathematics and Argumentation
Andrew Aberdein Found Sci (2009) 14:1–8



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So it looks to me as though it would be disastrous to take the uniform distribution over lines with some fixed number of wildcards (unless, perhaps, one had done some more preprocessing to get a stronger property than mere richness).

Here's an attempt to throw the spanner in the works of #32.

Frontstage mathematics

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Backstage mathematics

31. Gil, a quick remark about Fourier expansions and the $k = 3$ case. I want to explain why I got stuck several years ago when I was trying to develop some kind of Fourier approach. Maybe with your deep knowledge of this kind of thing you can get me unstuck again.

...hat I didn't explain myself clearly

1. A quick question. Furstenberg and Katznelson used the Carlson–Simpson theorem in their proof. Does anyone know that proof well enough to know whether the Carlson–Simpson theorem might play a role here? If so, I could add

33. In response to Ryan's ... enough. Here's what I meant.

This looks weird enough that it's probably a wrong idea, but I still feel there may be a "natural" probabilistic way to do it.

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So it looks to me as though it would be disastrous to take the uniform distribution over lines with some fixed number of wildcards (unless, perhaps, one had done some more preprocessing to get a stronger property than mere richness).

Connecting to existing frameworks

- Peirce's three types of reasoning
- Walton *et al's* argumentation schemes
- Toulmin's layout

Peirce’s three types of reasoning

1. **Deduction**

Rule All the beans from this bag are white. Case These beans are from this bag.
∴ Result These beans are white.

Peirce’s three types of reasoning

2. **Induction**

Case These beans are [randomly selected] from this bag. Result These beans are white.
∴ Rule All the beans from this bag are white.

Peirce's three types of reasoning

3. **Hypothesis [Abduction]**

Rule All the beans from this bag are white.
Result These beans [oddly] are white.
∴ Case These beans are from this bag.

The cornerstone of all scientific discovery

The surprising fact C is observed;
But if A were true, C would be a matter of course;
Hence, there is reason to suspect that A is true. [Peirce, 58, 5.188–89]

Two tasks:

- generation of different hypotheses
- selection of best hypothesis (to start testing)

“Every single item of scientific theory which stands established today has been due to Abduction.” [Peirce, 58, 8.172]

The central problem of abduction

Understanding the criteria for selection of the best hypothesis. It must:

1. explain the surprising fact

2. be subject to experimental testing

3. be economical (worth our time to investigate). We should consider:
   (a) cost of verifying/falsifying the hypothesis (should be low);

   (b) intrinsic value in the hypothesis (should be high)— value is (i) its simplicity—following Ockham's razor; and (ii) likelihood of it being true (estimated by previous experience).

   (c) the effect of the hypothesis on other projects

Deduction in maths...

Induction in maths...

Polya and Induction

Polyhedron	*F*	*V*	*E*
I. cube	6	8	12
II. triangular prism	5	6	9
III. pentagonal prism	7	10	15
IV. square pyramid	5	5	8
V. triangular pyramid	4	4	6
VI. pentagonal pyramid	6	6	10
VII. octahedron	8	6	12
VIII. "tower"	9	9	16
IX. "truncated cube"	7	10	15

Polya and Induction

Polyhedron	*F*	*V*	*E*
triangular pyramid . .	4	4	6
square pyramid . .	5	5	8
triangular prism . .	5	6	9
pentagonal pyramid .	6	6	10
cube	6	8	12
octahedron . . .	8	6	12
pentagonal prism . .	7	10	15
"truncated cube" . .	7	10	15
"tower" . . .	9	9	16

Polya and Induction

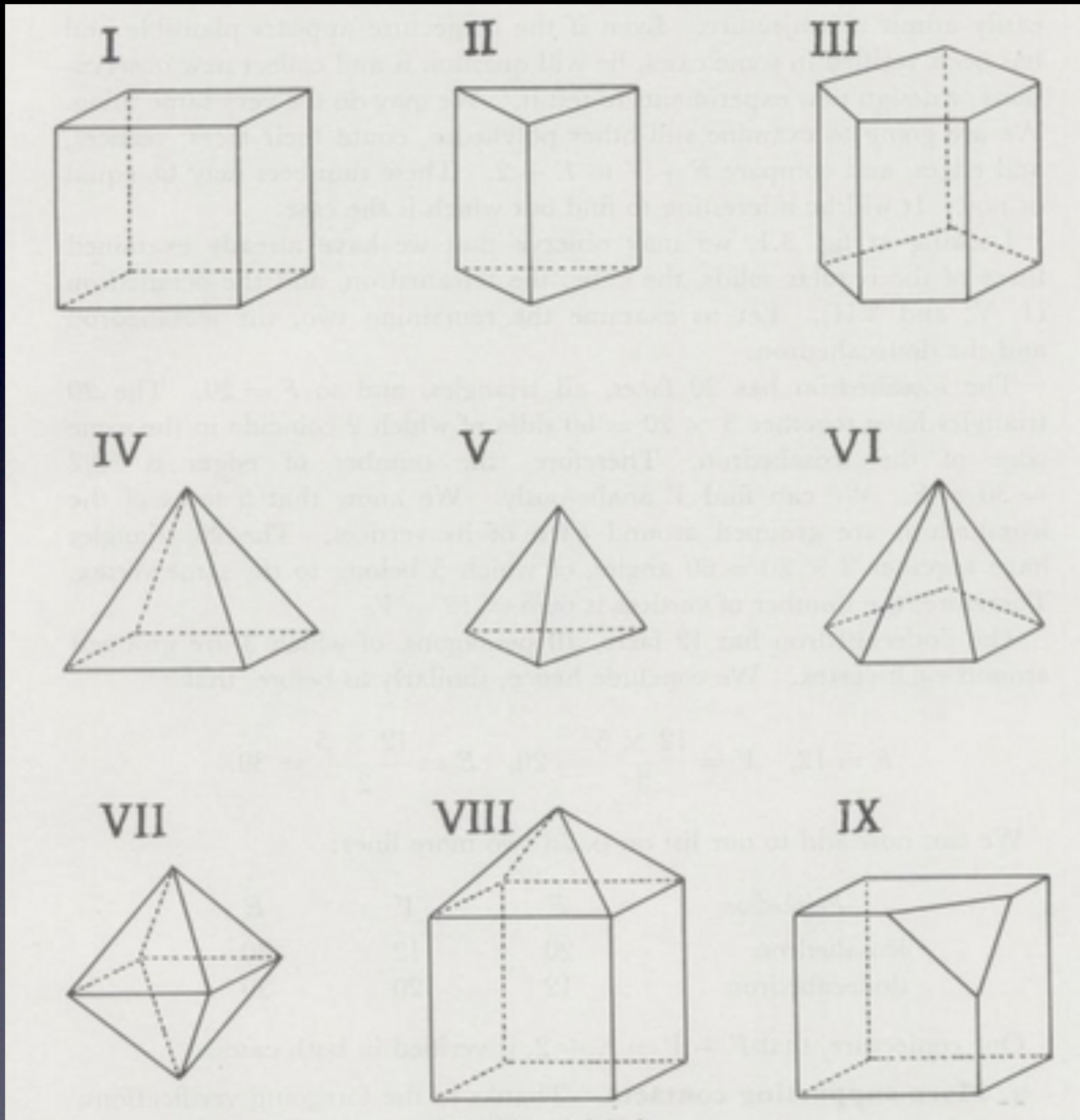
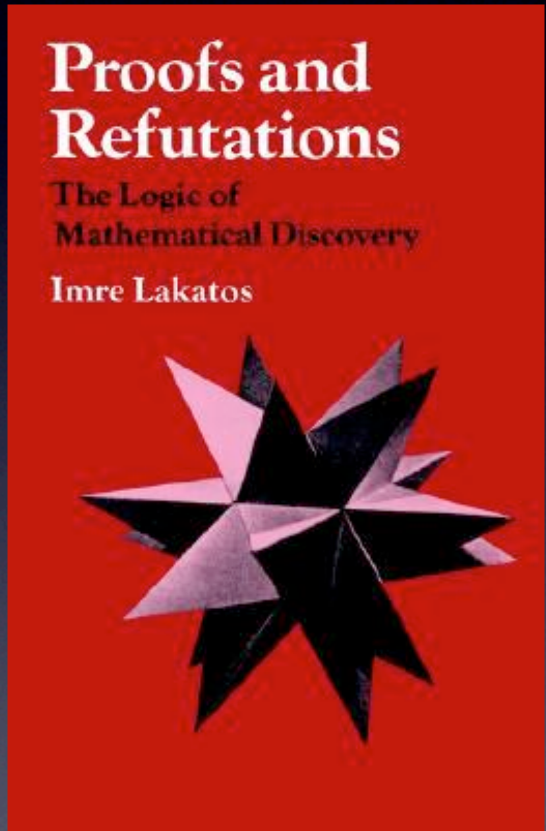


Fig. 3.1. Polyhedra.

Abduction in maths...

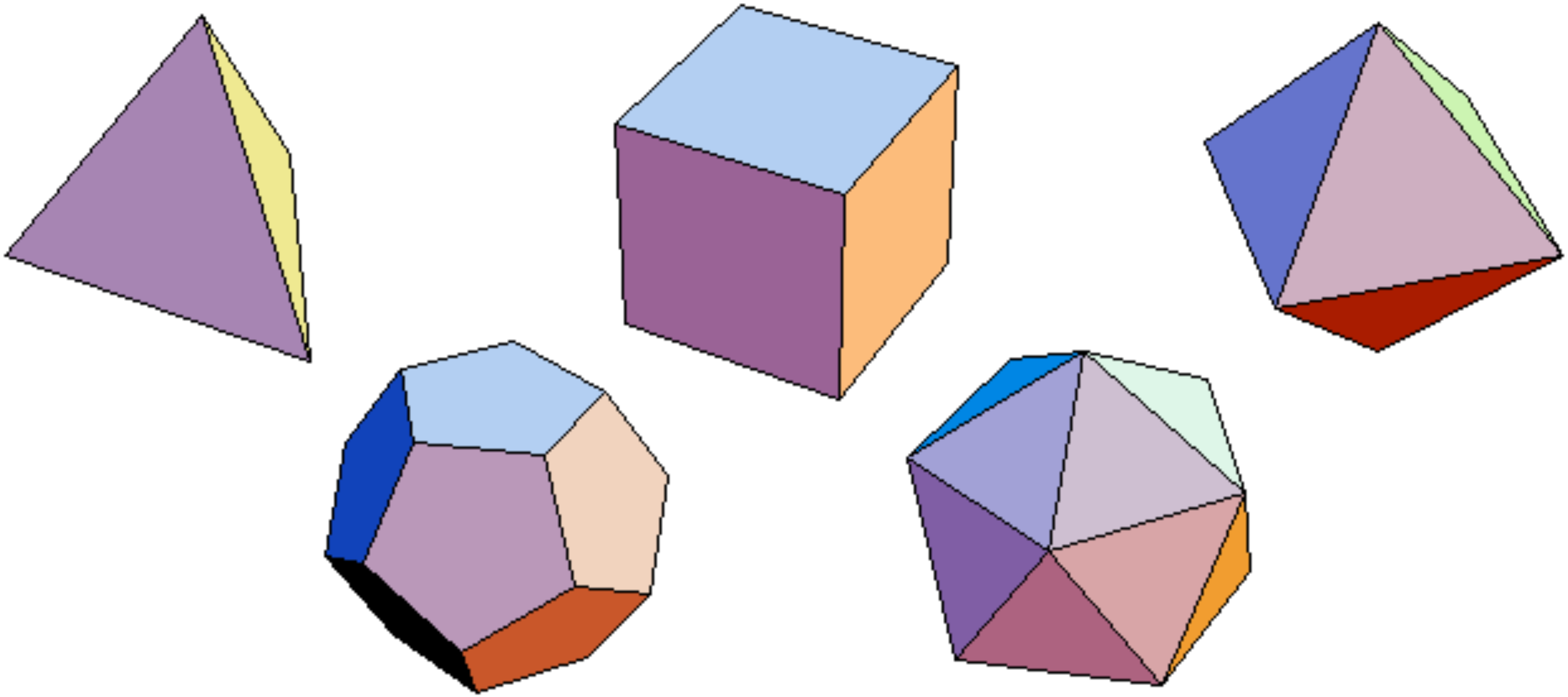


Lakatos's theory of

- Discussed the evolution of one particular argument in research mathematics over 200 years.
- Showed how concepts, conclusion and premises underwent change.
- Focused on the role that counterexamples played.

Claim

For any polyhedron, the number of vertices (V) minus the number of edges (E) plus the number of faces (F) = 2.



Argument that V - E + F = 2

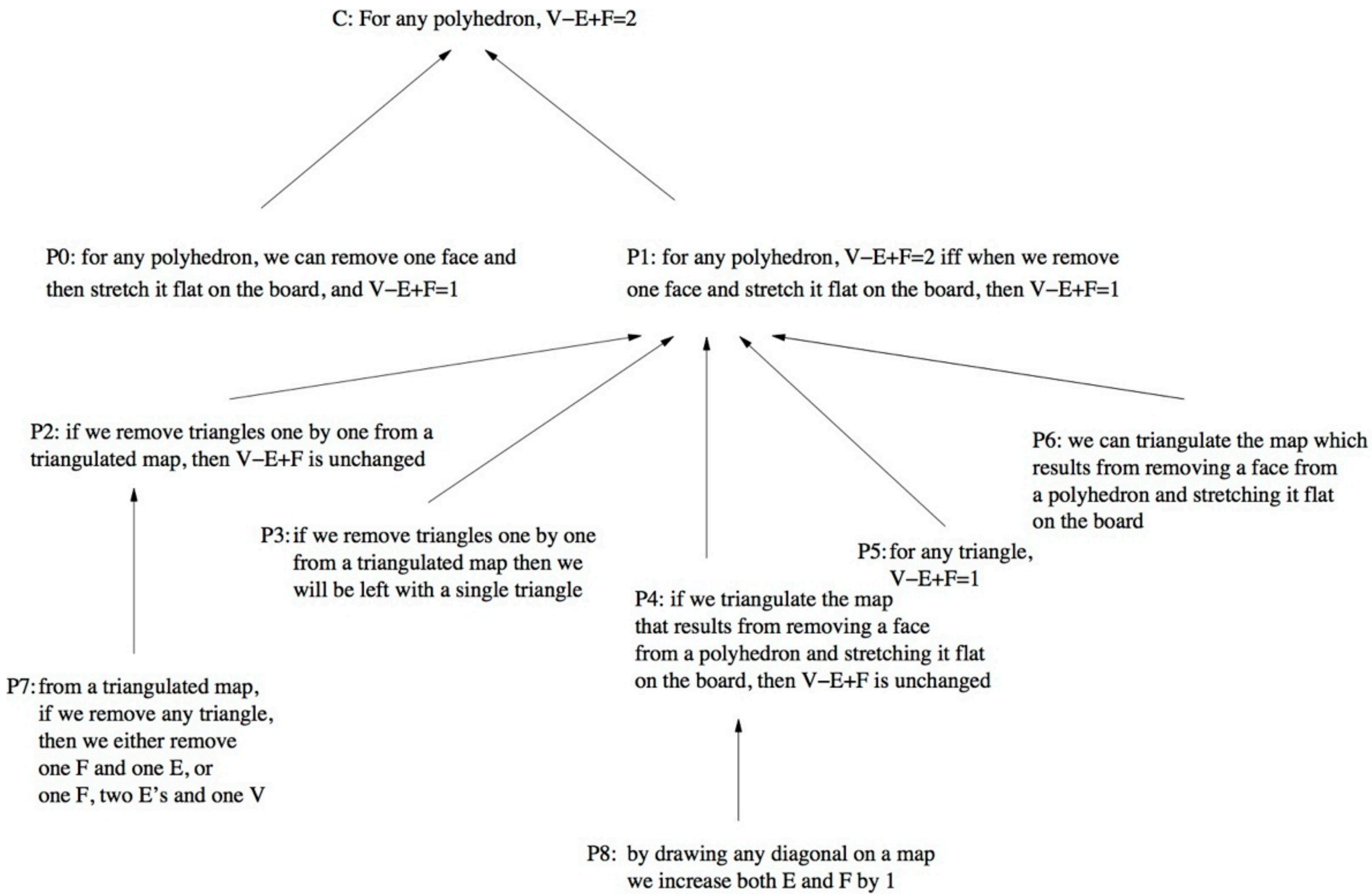
Step 1: Let us imagine the polyhedron to be hollow, with a surface made of thin rubber. If we cut one of the faces, we can stretch the remaining surfaces flat on the blackboard, without tearing it. The faces and edges will be deformed, the edges may become curved,and V and E will not alter, so that if and only if V-E+F= 2 for the original polyhedron, $V - E + F = 1$ for this flat network - remember that we have removed one face.

Argument that V - E + F = 2

Step 2: Now we triangulate our map - it does indeed look like a geographical map. We draw (possibly curvilinear) diagonals in those (possibly curvilinear) polygons which are not already (possibly curvilinear) triangles. By drawing each diagonal we increase both E and F by one, so that the total $V - E + F$ will not be altered.

Argument that $V - E + F = 2$

Step 3: From the triangulated map we now remove the triangles one by one. To remove a triangle we either remove an edge - upon which one face and one edge disappear, or we remove two edges and a vertex - upon which one face, two edges and and one vertex disappear. Thus, if we had $V - E + F = 1$ before a triangle is removed, it remains so after the triangle is removed. At the end of this procedure we get a single triangle. For this $V - E + F = 1$ holds true.


C: For any polyhedron, V−E+F=2
P0: for any polyhedron, we can remove one face and then stretch it flat on the board, and V−E+F=1
P1: for any polyhedron, V−E+F=2 iff when we remove one face and stretch it flat on the board, then V−E+F=1
P2: if we remove triangles one by one from a triangulated map, then V−E+F is unchanged
P3: if we remove triangles one by one from a triangulated map then we will be left with a single triangle
P4: if we triangulate the map that results from removing a face from a polyhedron and stretching it flat on the board, then V−E+F is unchanged
P5: for any triangle, V−E+F=1
P6: we can triangulate the map which results from removing a face from a polyhedron and stretching it flat on the board
P7: from a triangulated map, if we remove any triangle, then we either remove one F and one E, or one F, two E's and one V
P8: by drawing any diagonal on a map we increase both E and F by 1

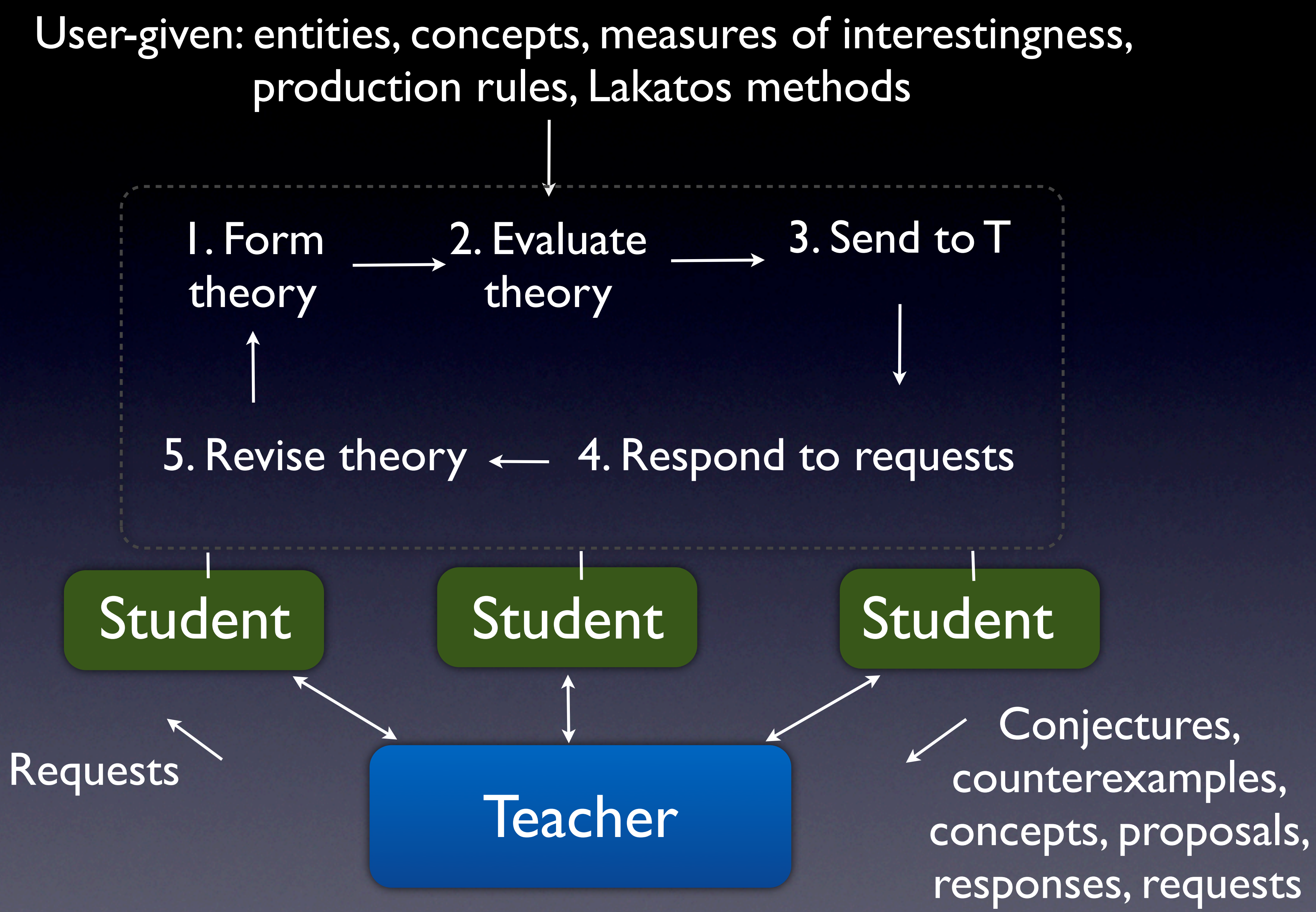
Challenge

Is the claim true?

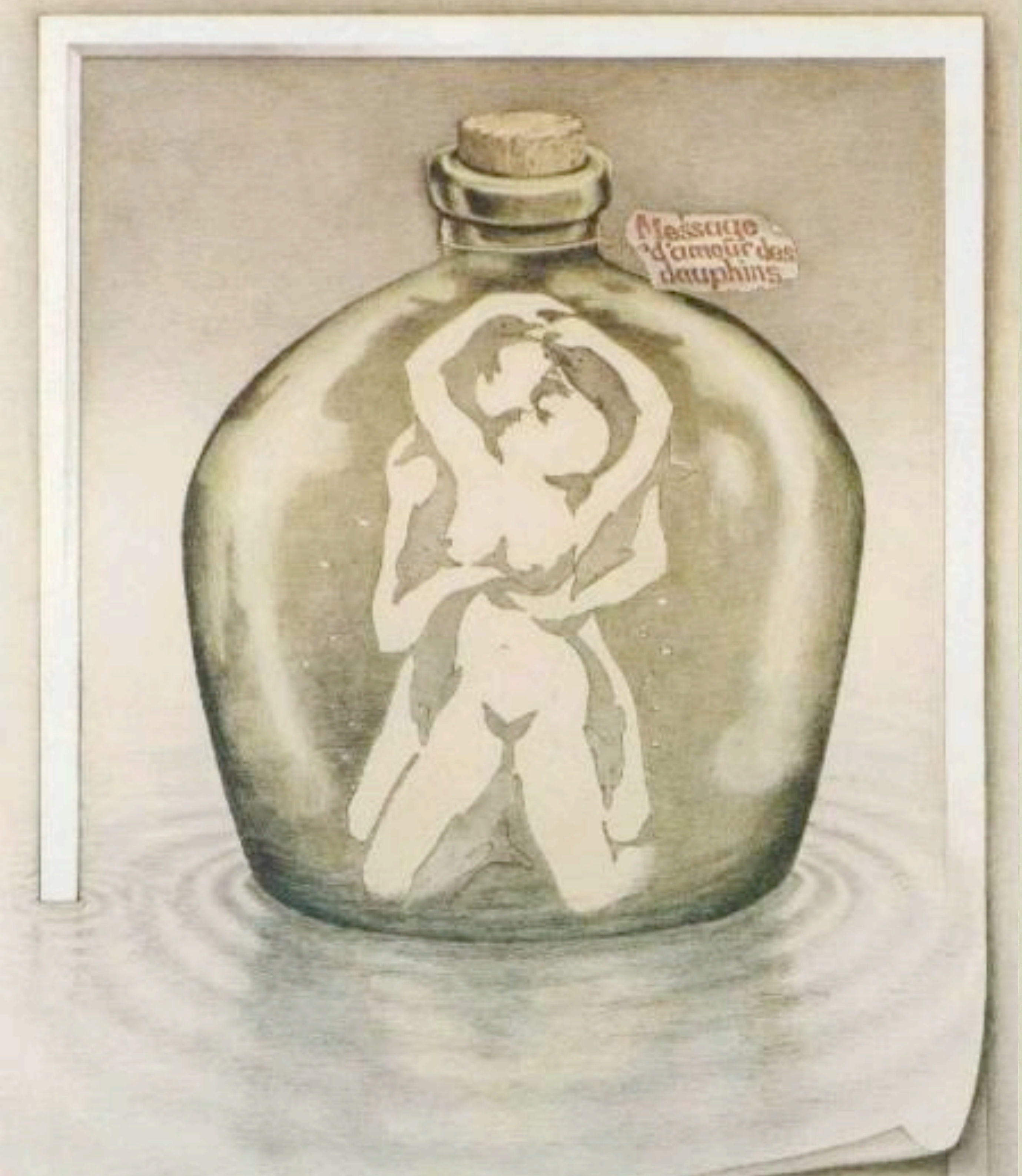
Is the argument valid?

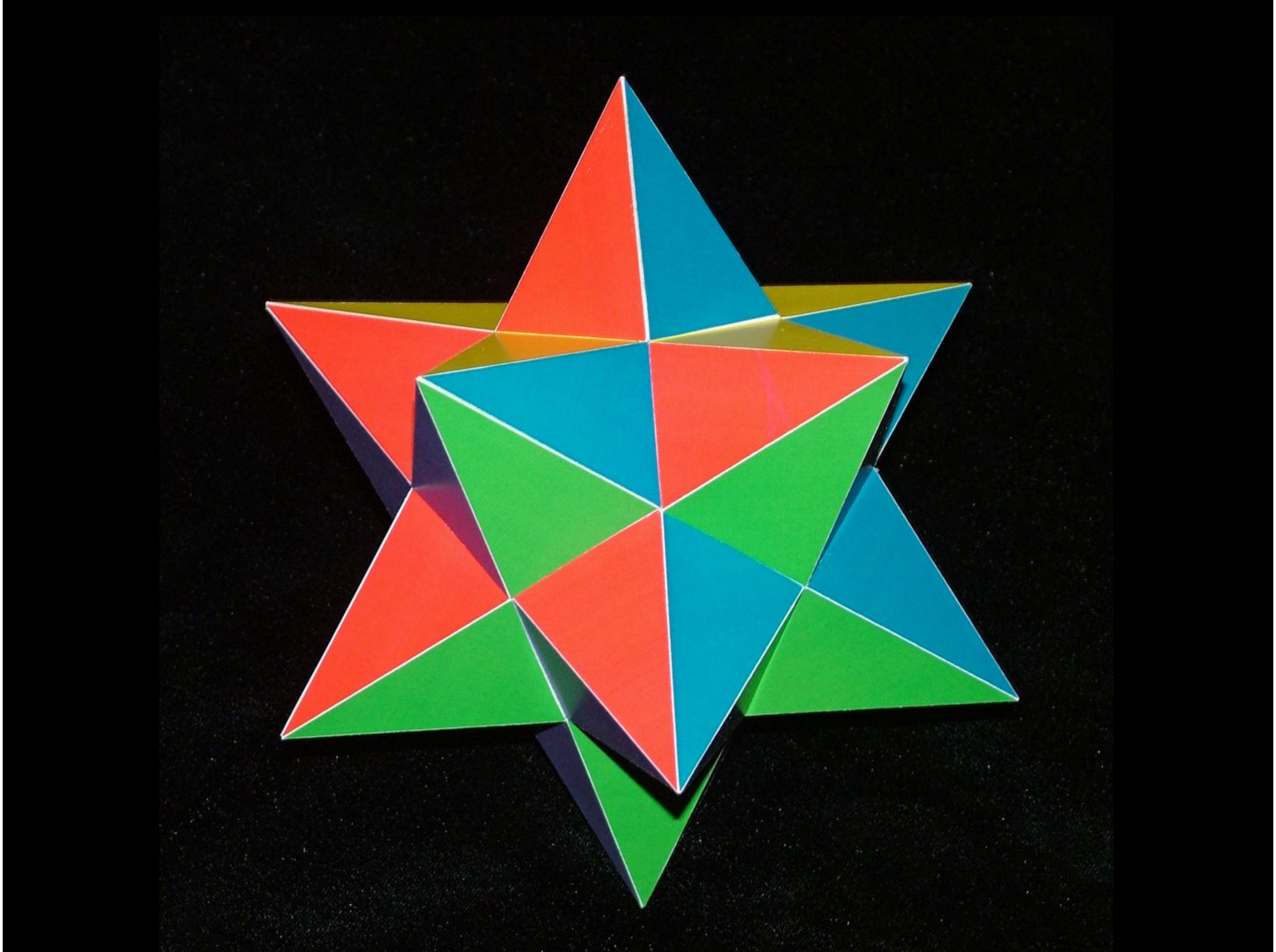
Responding to

1. *Monster-barring/adjusting:* (Re)define your terms in a way which excludes the counterexample.
2. *Exception-barring:* Exclude an object or class of objects from the conclusion.
3. *Lemma incorporation:* Find the (possibly missing) faulty premise in the argument, and incorporate this premise as a condition in the conclusion.

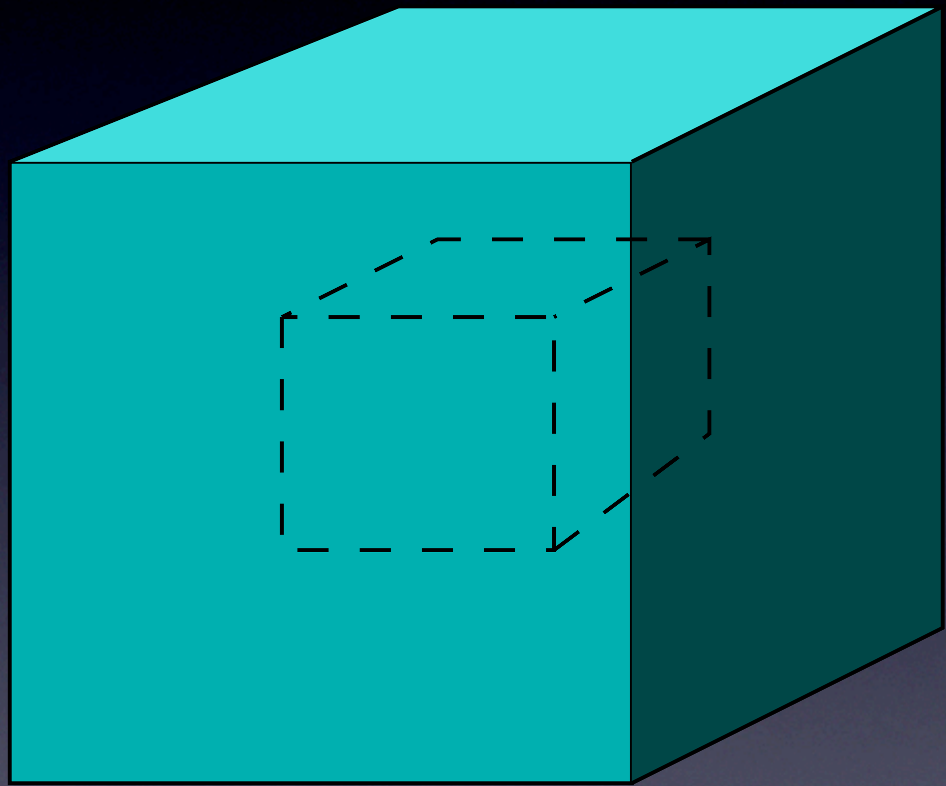

User-given: entities, concepts, measures of interestingness, production rules, Lakatos methods
1. Form theory
2. Evaluate theory
3. Send to T
4. Respond to requests
5. Revise theory
Student
Student
Student
Requests
Teacher
Conjectures, counterexamples, concepts, proposals, responses, requests

Evolving concepts


Message
d'amour des
dauphins

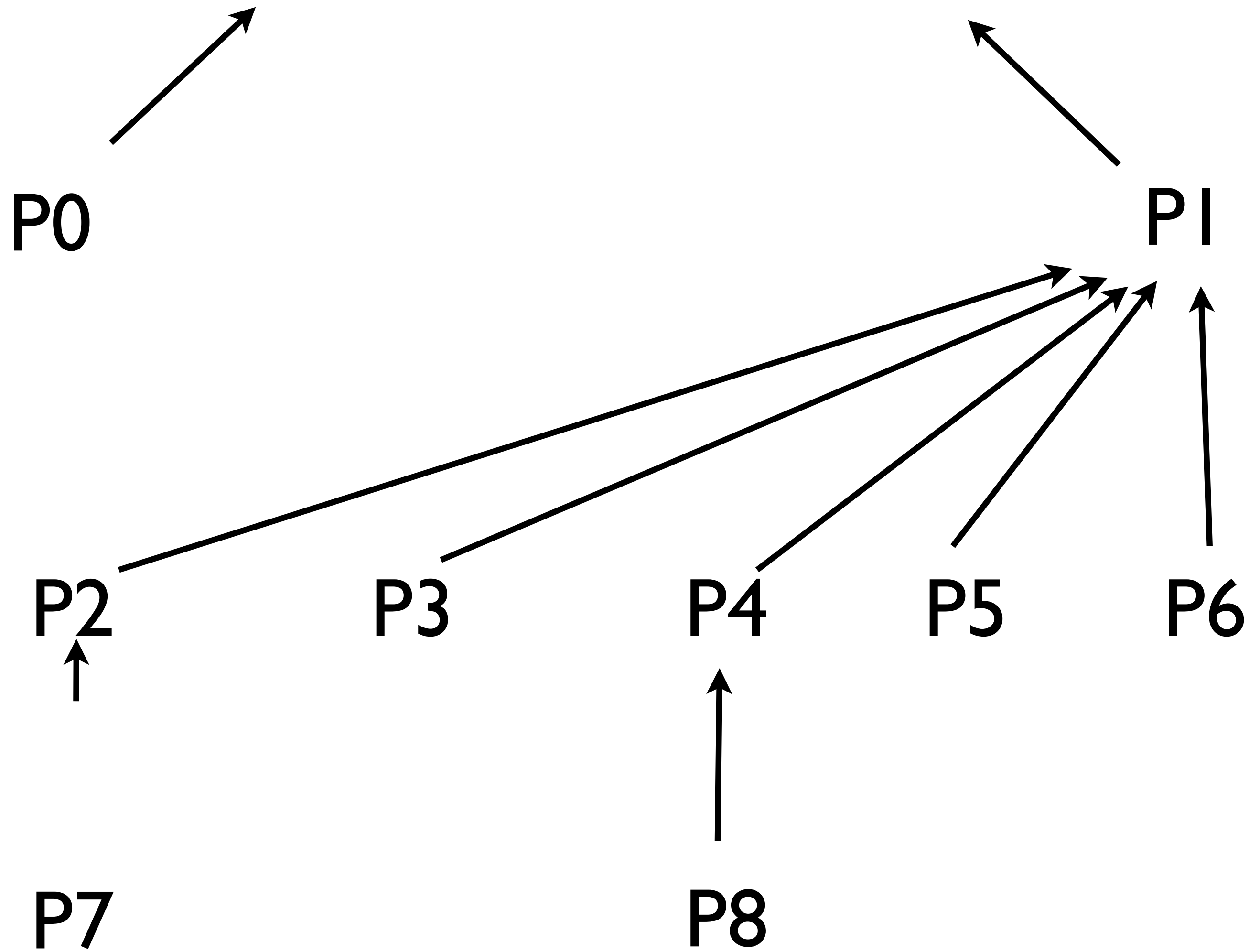


The hollow cube



16 - 24 + 12 = 4

C: For any polyhedron, $V - E + F = 2$



[1-10,integer, div,mult]

There do not exist integers a, b such that

$b + a = a$ *and* $a + b = a$

[1-10,integer, div,mult]
There do not exist integers a, b such that
b + a = a and a + b = a

[0-10,integer, div,mult]
zero and any other integer

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[0-10,integer, div,mult]
zero and any other integer

Zero is a problem entity.
I suggest we monster-bar it.

[1-10,integer, div,mult]
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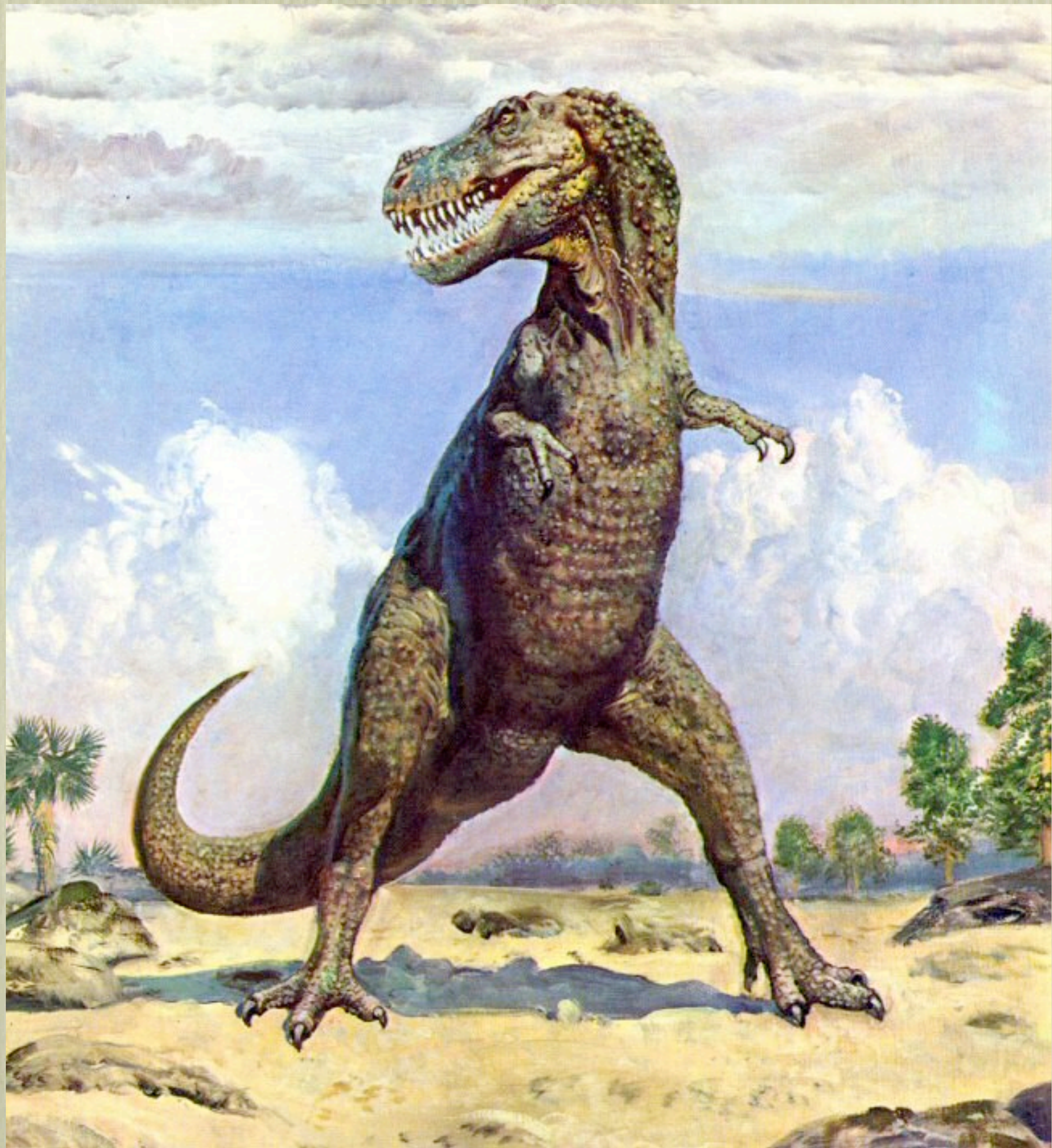
[Checks new object against current theory.
Finds it breaks 63% of its conjectures]
Okay - I'll accept that.

Rogue taxidermy and the platypus



There does not exist an animal which produces milk and lays eggs.

There does not exist an animal which produces milk and lays eggs.

The platypus does.

There does not exist an animal which produces milk and lays eggs.

The platypus does.

[Checks new object against current theory. Finds it breaks 11% of its conjectures]
The platypus is not an animal

There does not exist an animal which produces milk and lays eggs.

The platypus does.

[Checks new object against current theory. Finds it breaks 11% of its conjectures]

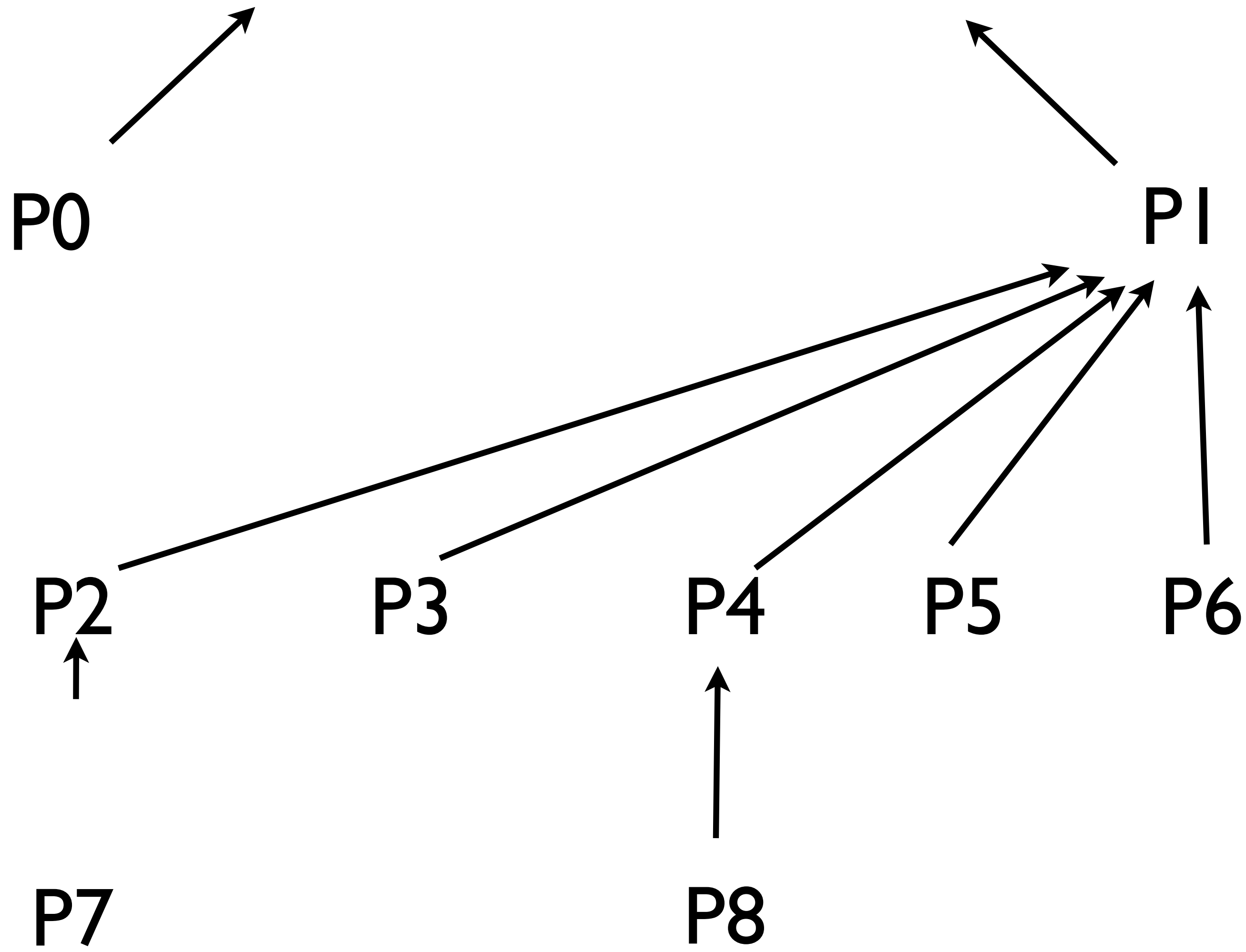
The platypus is not an animal

[Finds that the platypus breaks 31% of its own conjectures.]

Okay - I'll accept that.

Evolving conclusions

C: For any polyhedron, except those with cavities, V - E + F = 2



1. Goldbach's conjecture:

All even numbers are the sum of two primes $\longrightarrow$

All even numbers except 2 *are the sum of two primes*

2. All groups are Abelian $\longrightarrow$

All self-inverse *groups are Abelian*

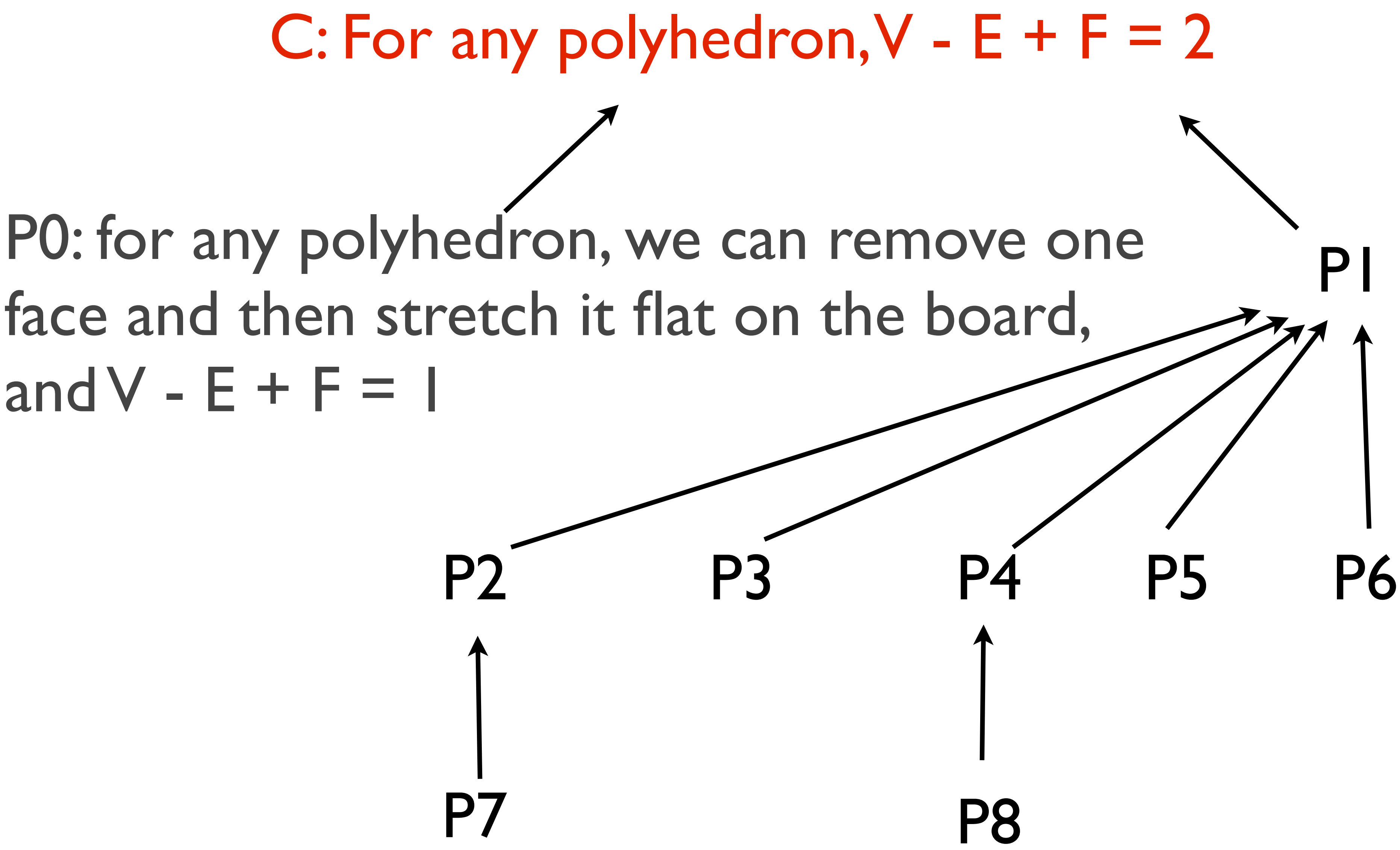
3. All integers have an even number of divisors $\longrightarrow$

All non-squares *have an even number of divisors*

From TPTP library we invented 91 non-theorems. TM produced valid modifications for 83\% of them, with an

Rise of the Robogeeks, Michael Brooks.
New Scientist 2697, March 3rd 2009

Evolving premises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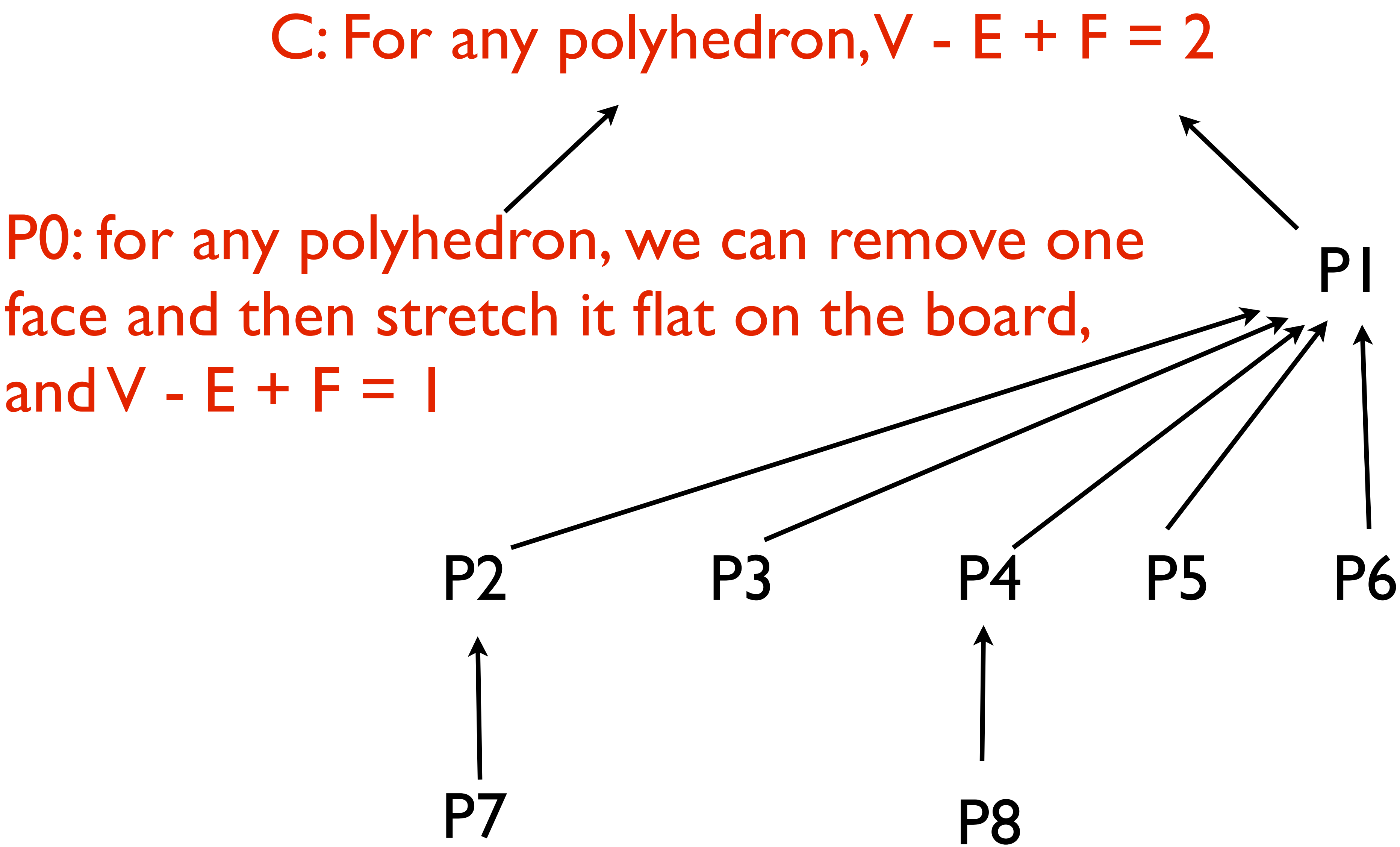
P0: for any polyhedron, we can remove one face and then stretch it flat on the board, and V - E + F = 1
P1
P2
P3
P4
P5
P6
P7
P8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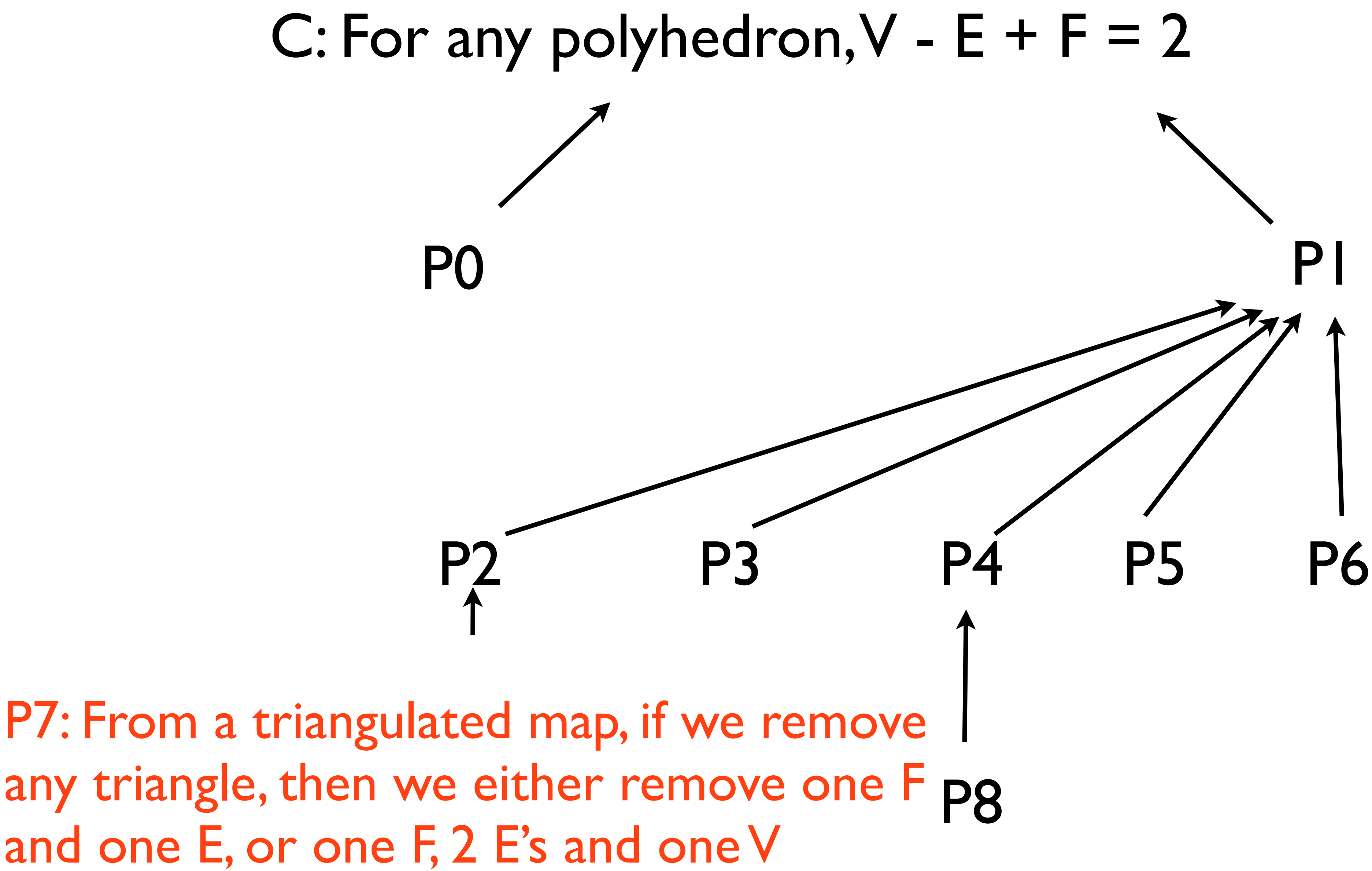
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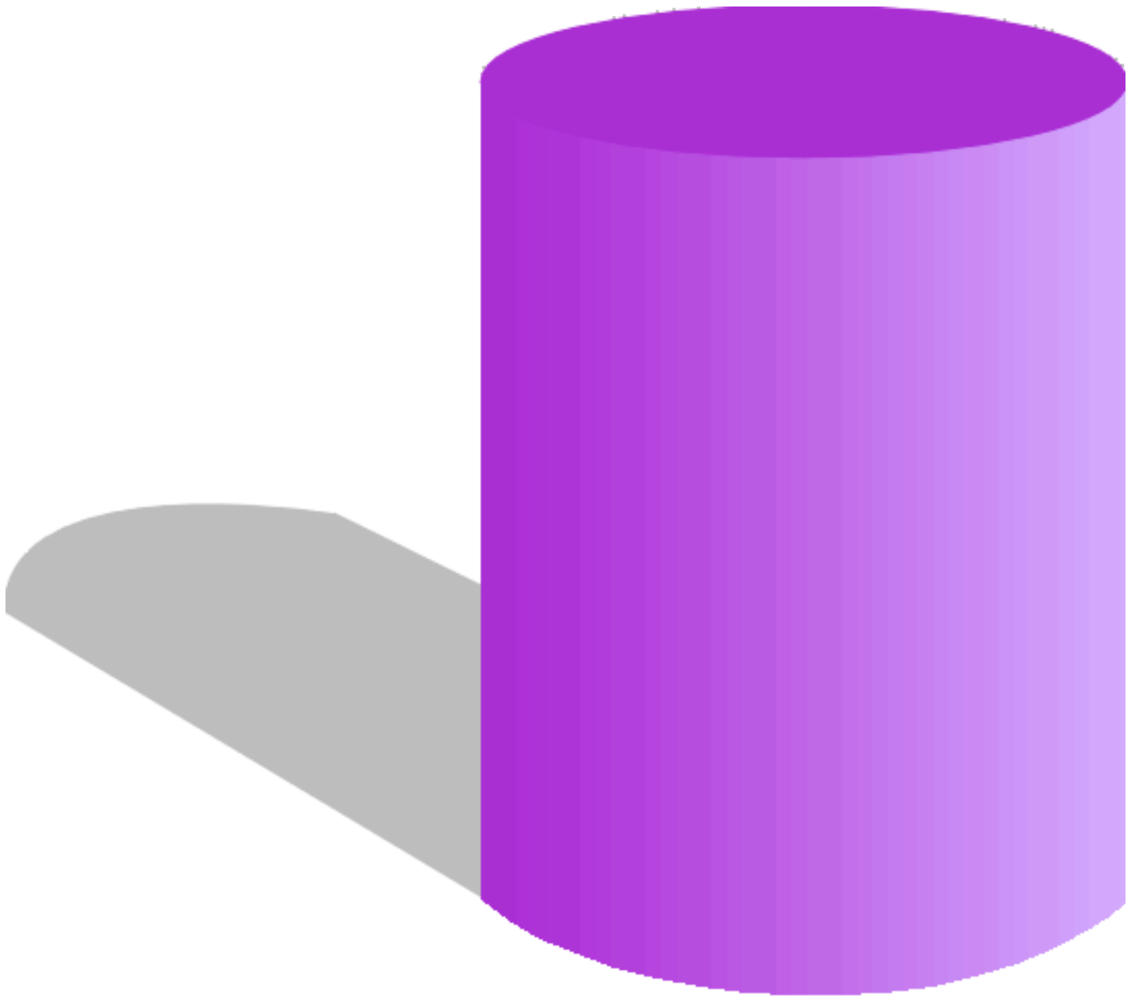
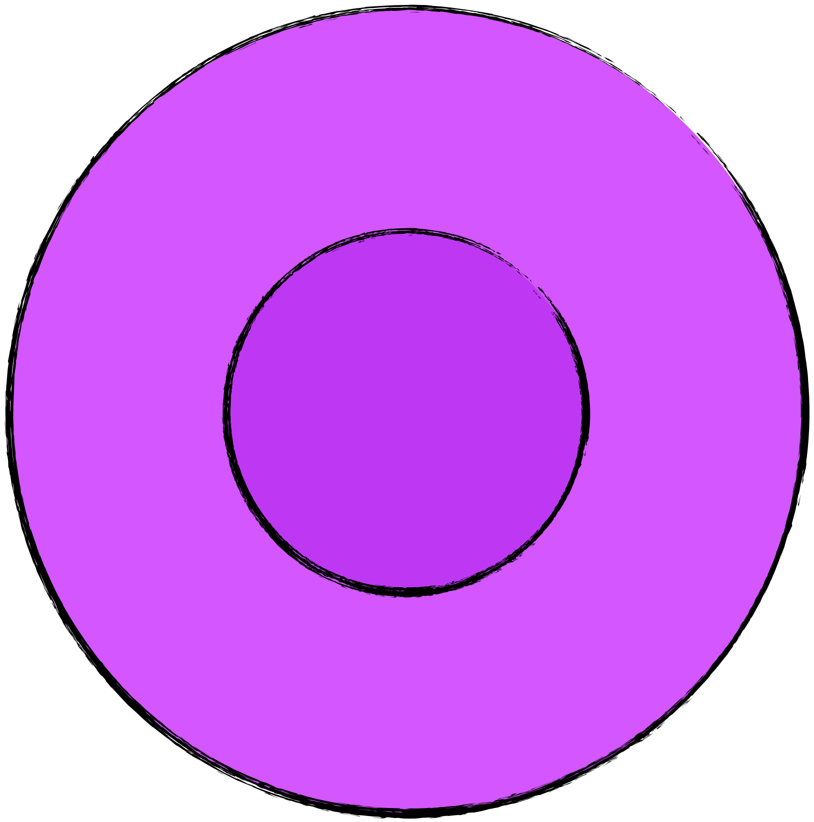
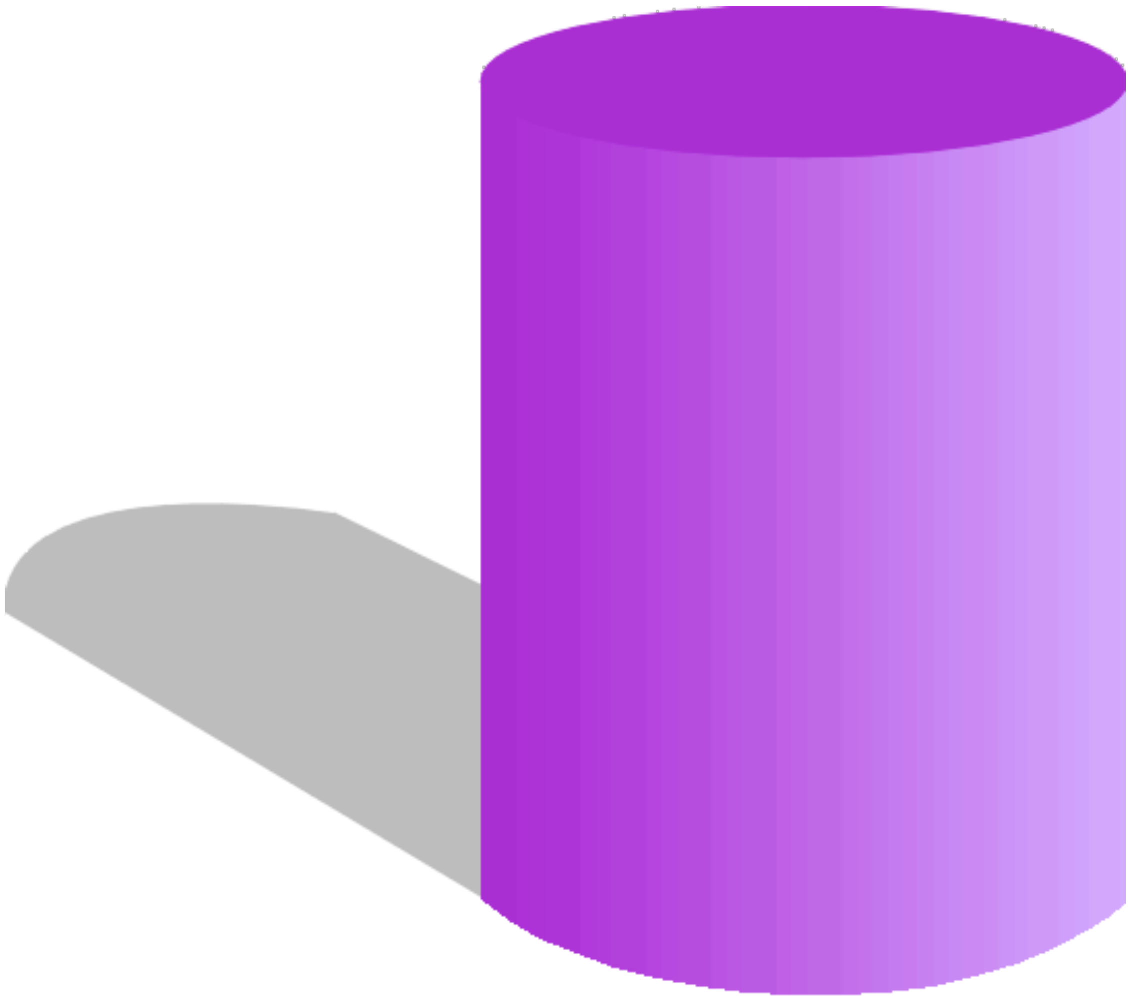
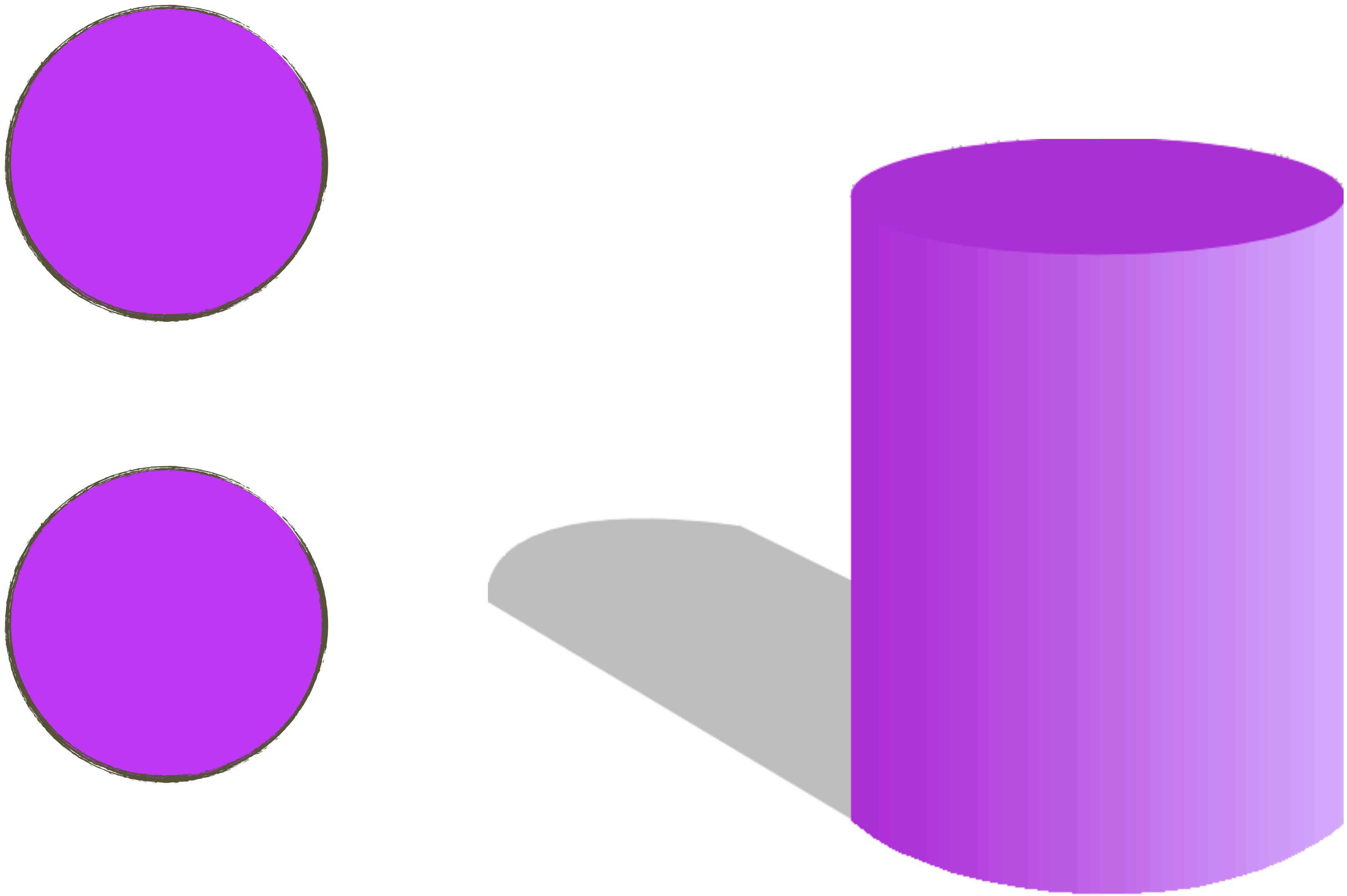
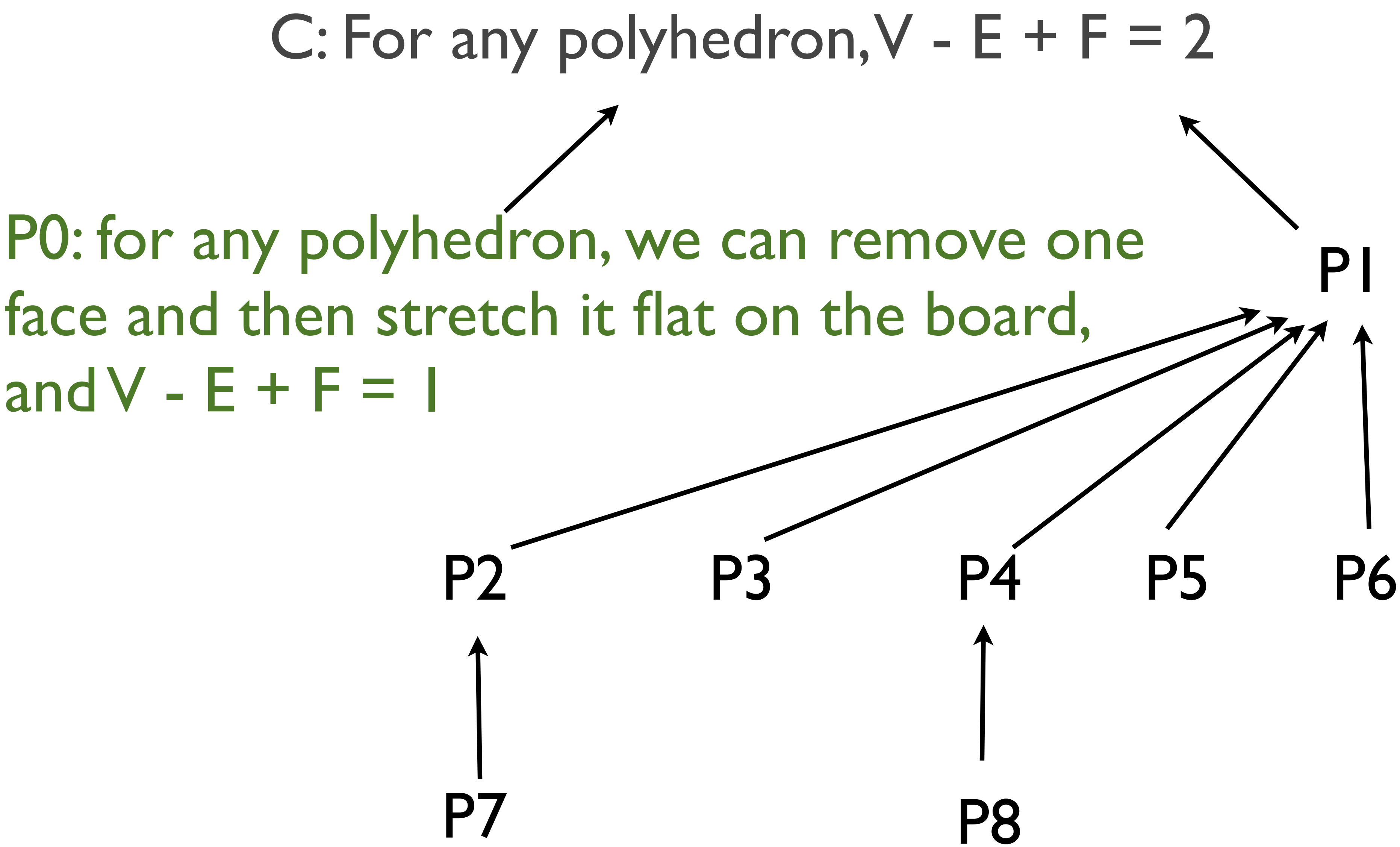
Cylinder:

$$V-E+F=0-2+3=1$$



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- **Part II:** philosophical and linguistic differences between the processes of constructing and presenting mathematical proof

Online discussion sites for mathematicians

The polymath blog

mathoverflow

In collaboration with Prof Ursula Martin and Associate Prof Andrew Aberdein

Online collaborative mathematics

- successful mathematical practice is characteristically collaborative
- increasing ubiquity and reliability of online networking tools has facilitated the growth of remote collaboration
- *‘These examples [Linux, Wikipedia, and a chess match between Kasparov and a “World Team”] are not curiosities, or special cases; they are just the leading edge of the greatest change in the creative process since the invention of writing’*

Nielsen, M. Reinventing Discovery: The New Era of Networked Science, Princeton University Press, USA, 2011.

Implications for the study of mathematical practice

- Online forums and blogs for informal mathematical discussion reveal some of the 'back' of mathematics:

  'mathematics as it appears among working mathematicians, in informal settings, told to one another in an office behind closed doors'

  Hersh, R. (1991). Mathematics has a front and a back. Synthese, 88:127–133.

- *'it has provided, for possibly the first time ever (though I may well be wrong about this), the first fully documented account of how a serious research problem was solved, complete with false starts, dead ends etc. interested'*

  Gowers, T. (2009). Polymath1 and open collaborative mathematics. http://gowers.wordpress.com/2009/03/10/.

2. Connecting the dots: At the point where the pivot changes we create a line that passes through the previous pivot and a new pivot – like a side of a polygon.

0 0 Rate This

Comment by Gal — July 19, 2011 @ 8:07 pm | Reply

Nice. We need only to consider the times when to points are connected – this gives us a path, and after some time this path will come back to some already visited point. So there is a cycle. If only we could find a cycle which spans all the points, the question is solved… That may be some useful simplification.

1 0 Rate This

Comment by Garf — July 19, 2011 @ 8:23 pm | Reply

Isn't there always a cycle that spans all the points? The problem imposes restrictions on the cycles we can choose, right?

1 0 Rate This

Comment by Gal — July 19, 2011 @ 8:37 pm | Reply

For example, the restriction on how the next pivot is chosen (geometrically: comment 9). Are there any other restrictions? Can we start with a complete graph and all cycles on that graph and just discard the ones that don't follow the restrictions to converge on the ones that do?

1 1 Rate This

Comment by Gal — July 19, 2011 @ 8:56 pm | Reply

The line must sweep out a full rotation (and only one full rotation) of 2π during the traversal of S. I feel like this is intimately related to proving that there is a starting angle for any point P in S such that all of S is then traversed. I'm trying to show this by induction. Base case (|S|=2) is obvious. Let |S| = n, take S' = S U {Q}, and start with some windmill traversal of S.

Case A: Q is unreachable. Therefore we just traverse S, taking 2π to do so by induction.

3. If the points form a convex polygon, it is easy.

0 1 Rate This

Comment by Anonymous — July 19, 2011 @ 8:08 pm | Reply

Yes. Can we do it if there is a single point not on the convex hull of the points?

1 0 Rate This

Comment by Thomas H — July 19, 2011 @ 8:09 pm | Reply

Say there are four points: an equilateral triangle, and then one point in the center of the triangle. No three points are collinear.
It seems to me that the windmill can not use the center point more than once! As soon as it hits one of the corner points, it will cycle indefinitely through the corners and never return to the center point.
I must be missing something here...

0 0 Rate This

Comment by Jerzy — July 19, 2011 @ 8:17 pm | Reply

This isn't true – it will alternate between the centre and each vertex of the triangle.

0 0 Rate This

Comment by Joe — July 19, 2011 @ 8:21 pm | Reply

No, you're not right. Let the corner points be A, B, C, clockwise, M the center. If you start in M, you first hit say A, then C, then M, then B, then A.

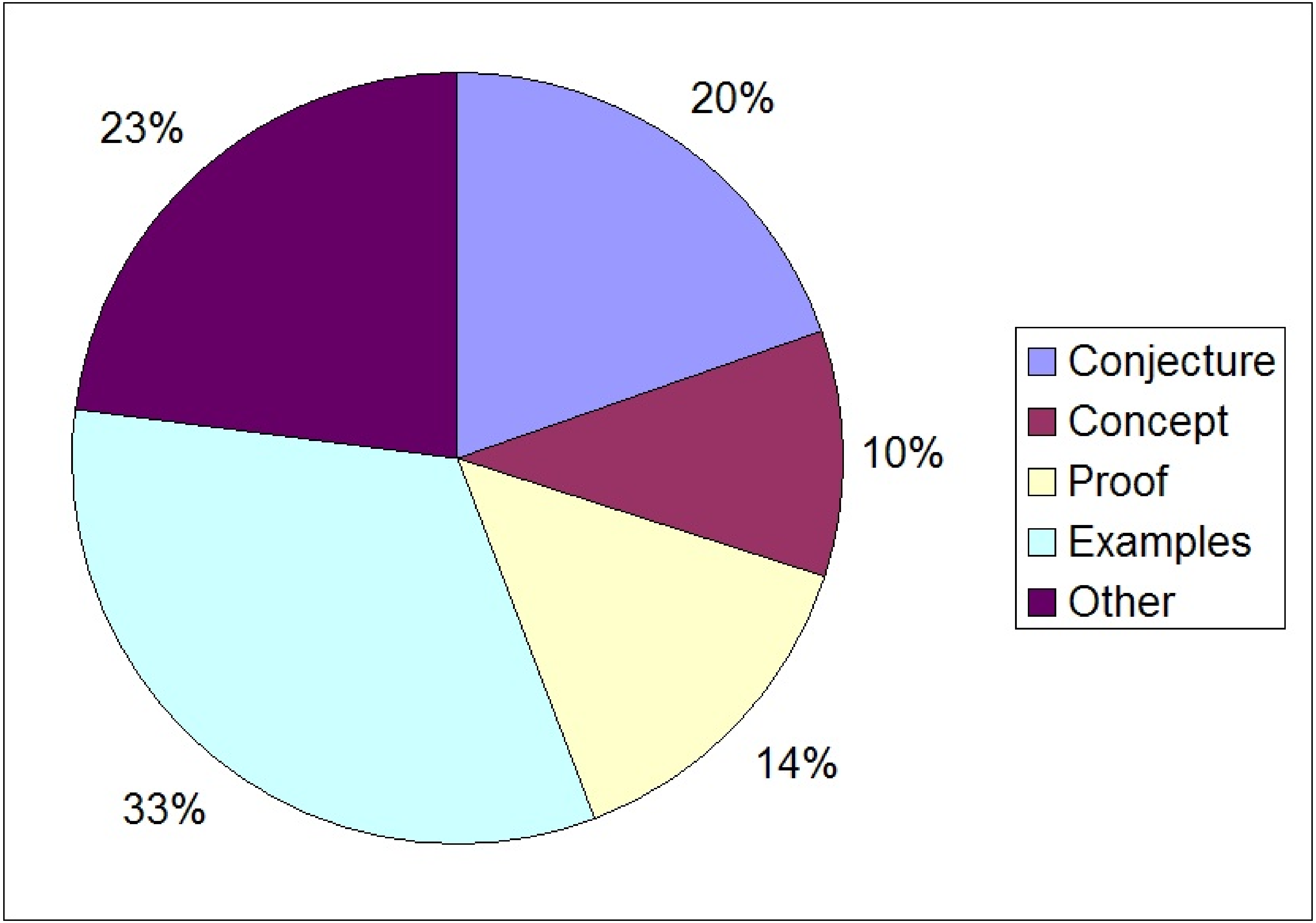
2 1 Rate This

Comment by Thomas H — July 19, 2011 @ 8:21 pm | Reply

Ohhh... I misunderstood the problem. I saw it as a half-line extending out from the last point, in which case you would get stuck on the convex hull. But apparently it means a full line, so that the next point can be "behind" the previous point. Got it.

1 0 Rate This

Comment by Jerzy — July 19, 2011 @ 8:31 pm | Reply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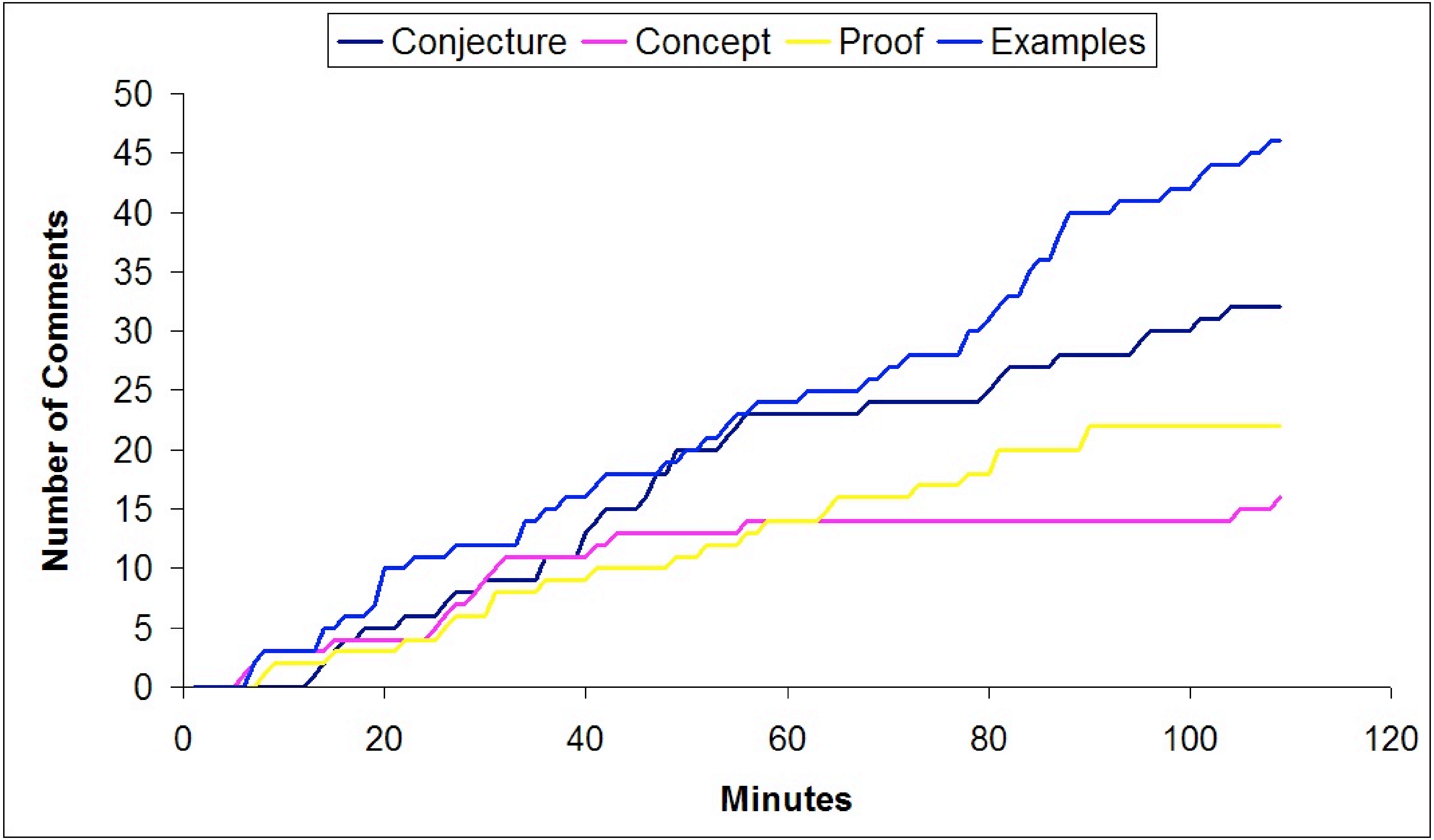
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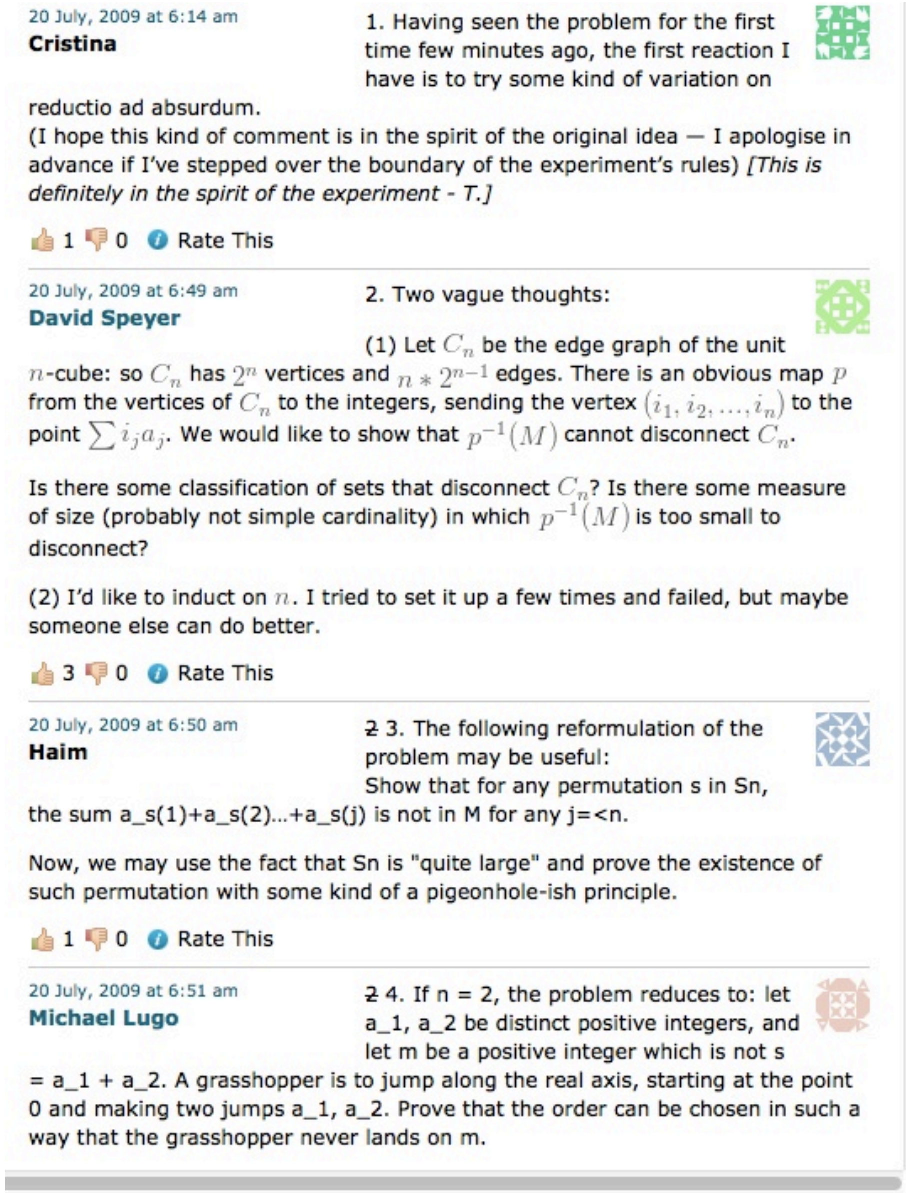

20 July, 2009 at 6:14 am
Cristina

1. Having seen the problem for the first time few minutes ago, the first reaction I have is to try some kind of variation on reductio ad absurdum.
(I hope this kind of comment is in the spirit of the original idea — I apologise in advance if I've stepped over the boundary of the experiment's rules) *[This is definitely in the spirit of the experiment - T.]*

1 0 Rate This

20 July, 2009 at 6:49 am
David Speyer

2. Two vague thoughts:

(1) Let C_n be the edge graph of the unit n-cube: so C_n has 2^n vertices and $n * 2^{n-1}$ edges. There is an obvious map p from the vertices of C_n to the integers, sending the vertex $(i_1, i_2, \ldots, i_n)$ to the point $\sum i_j a_j$. We would like to show that $p^{-1}(M)$ cannot disconnect C_n.

Is there some classification of sets that disconnect C_n? Is there some measure of size (probably not simple cardinality) in which $p^{-1}(M)$ is too small to disconnect?

(2) I'd like to induct on n. I tried to set it up a few times and failed, but maybe someone else can do better.

3 0 Rate This

20 July, 2009 at 6:50 am
Haim

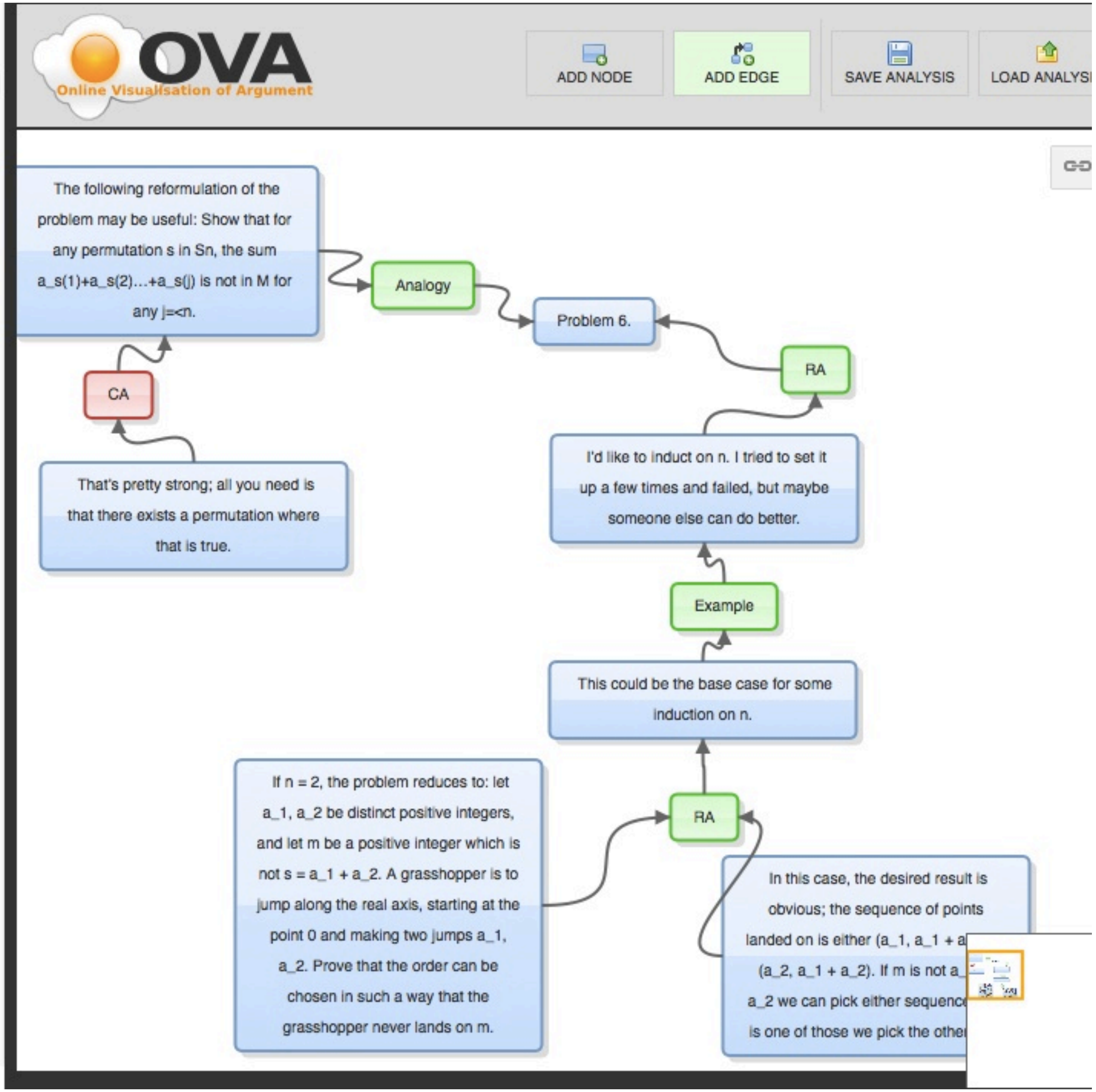
~~2~~ 3. The following reformulation of the problem may be useful:
Show that for any permutation s in Sn, the sum a_s(1)+a_s(2)…+a_s(j) is not in M for any j=<n.

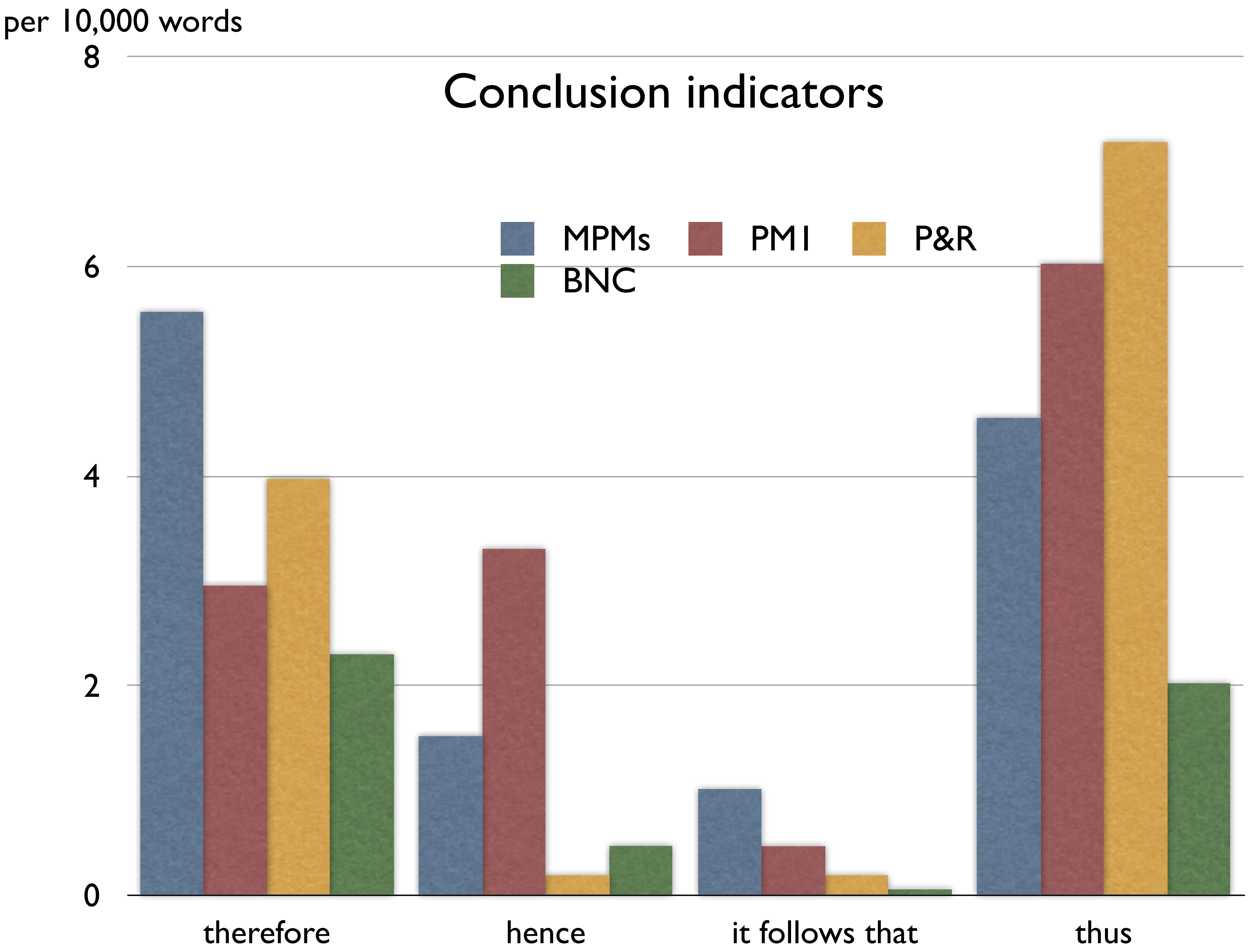
Now, we may use the fact that Sn is "quite large" and prove the existence of such permutation with some kind of a pigeonhole-ish principle.

1 0 Rate This

20 July, 2009 at 6:51 am
Michael Lugo

~~2~~ 4. If n = 2, the problem reduces to: let a_1, a_2 be distinct positive integers, and let m be a positive integer which is not s = a_1 + a_2. A grasshopper is to jump along the real axis, starting at the point 0 and making two jumps a_1, a_2. Prove that the order can be chosen in such a way that the grasshopper never lands on m.

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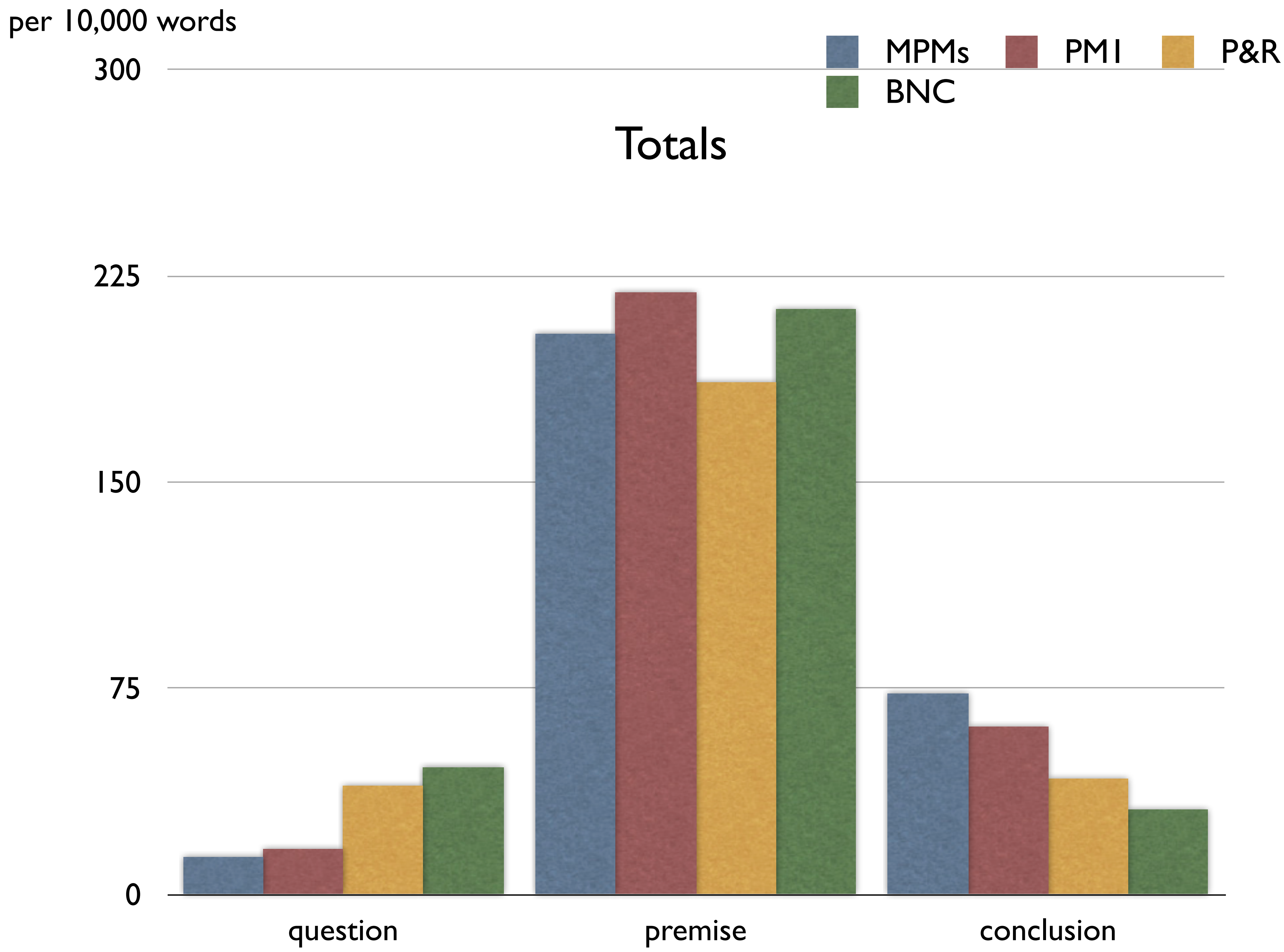
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conclusion

Walton’s schemes

Analogy

- Generally, case C1 is similar to case C2.
- A is true (or false) in C1.
- Therefore, A is true (or false) in C2.

Analogy

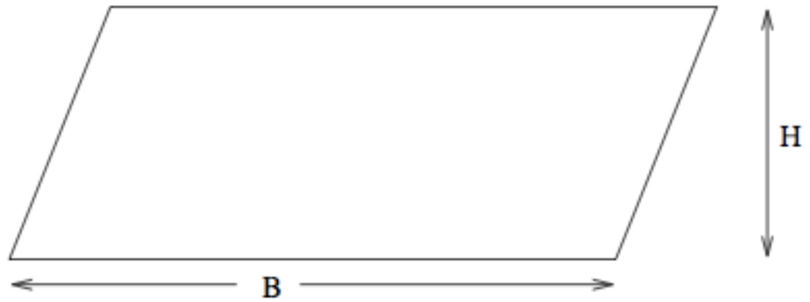
Critical questions

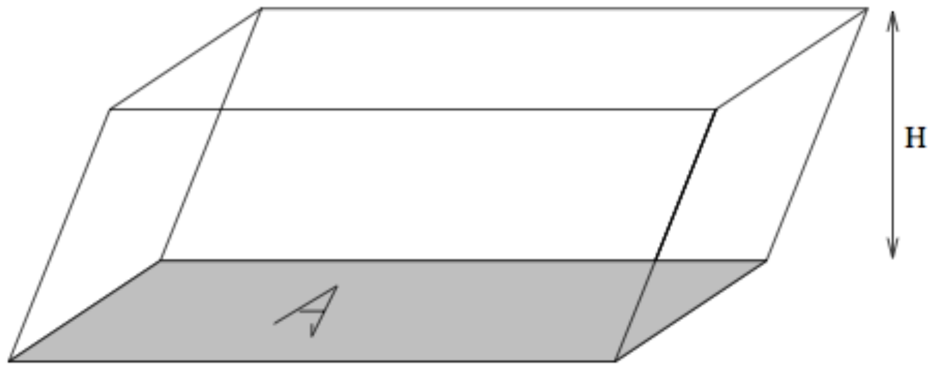
1. Are C1 and C2 similar, in the respect cited?
2. Is A true (false) in C1?
3. Are there differences between C1 and C2 that would tend to undermine the force of the similarity cited?
4. Is there some other case C3 that is also similar to C1, but in which A is false (true)?

Analogy between two and three dimensions

line	$\longrightarrow$	plane
length	$\longrightarrow$	area
area	$\longrightarrow$	volume
polygon	$\longrightarrow$	polyhedron

An inference which holds...

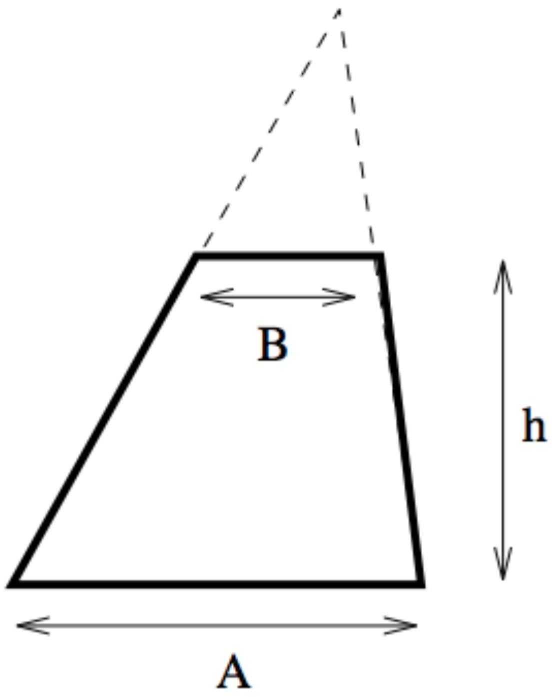


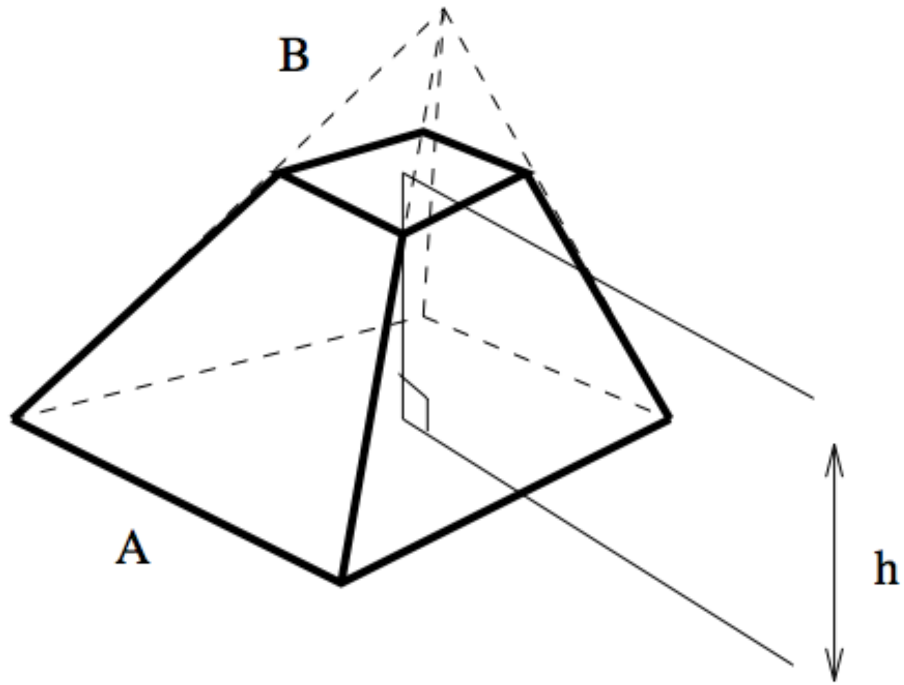


Area = B*H

Volume = Area*H

...and one which doesn't





Popularity

- If a large majority (everyone, nearly everyone, etc) accept A as true, then there exists a (defeasible) presumption in favour of A.
- A majority accept A as true
- Therefore, there exists a presumption in favour of A

Popularity

Critical questions:

1. Does a large majority accept A as true?
2. Is there other relevant evidence which would support the assumption that A is not true?
3. What reason is there for thinking that this large majority is right?

There is no Algebraist nor Mathematician so expert in his science, as to place entire confidence in any truth immediately upon his discovery of it, or regard it as any thing, but a mere probability. Every time he runs over his proofs, his confidence encreases; but still more by the approbation of his friends; and is raisd to its utmost perfection by the universal assent and applauses of the learned world.

Hume *Treatise on Human Nature,* 1739

Ongoing work...

People	Research Question	Methodology
Lakatos	Is there a logic of discovery and justification?	Historical/philosophical analysis, rational reconstruction
Alan Smaill, Simon Colton, John Lee, Alison Pease	Is it possible/useful to write a computational representation of Lakatos?	Implement and evaluate: interpret/extend/test
Lakoff and Nunez	How do new concepts arise in maths?	Linguistic analysis
Goguen	How do new concepts arise in maths?	Logical analysis

People	Research Question	Methodology
Alan Smaill, Markus Guhe, Dan Winterstien, Ewen Maclean, Alison Pease, Joe Corneli, ...	Is it possible/useful to write a computational representation of concept-blending/ metaphors?	Implement and evaluate: interpret/extend/test
Ursula Martin, Andrew, Aberdein, Alison Pease	What are people talking about? How does explanation work in maths?	Qualitative: data-driven ("based on GT" - dedoose, 74) and hypothesis-driven (was Lakatos right?; explanation: understanding how/that; implicit why questions, ...)

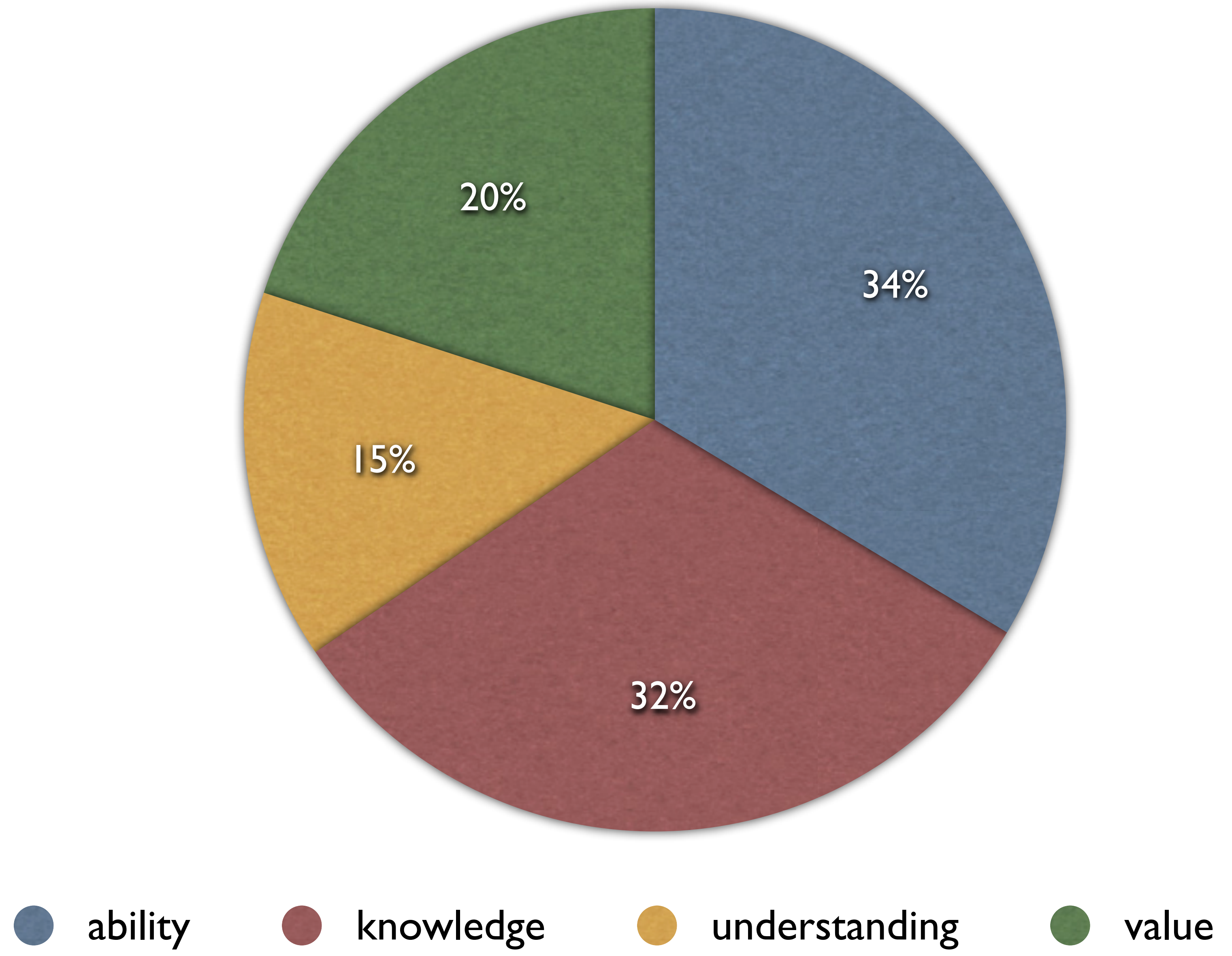
People-centred:

1. **Abilities (what can/can't we do):** [difficulty, hard, do] We can only almost do P; We can do X; We must be able to do X; X is always possible; X is possible; We can reduce prob to P; C might not be hardest bit; We can fix problem in this way; we can do Y; The difficult bit might be H
2. **knowledge (what do/don't we know):** [know, plausible, mistake, wrong] We don't know X; X is plausible; X is wrong; this is a mistake
3. **understand (what do/don't we understand):** [understand,] Why is this a contradiction?
4. **Value/goals (what do/don't we want):** [want, goal, need, help, problem] X is a good idea; We want to do X (is this proof?); X will achieve our goal Y; We need to know X; It will help us in this way; This problem might happen; This problem won't happen; Our solution might not always work; This cannot happen; P might happen; What if this problem occurs

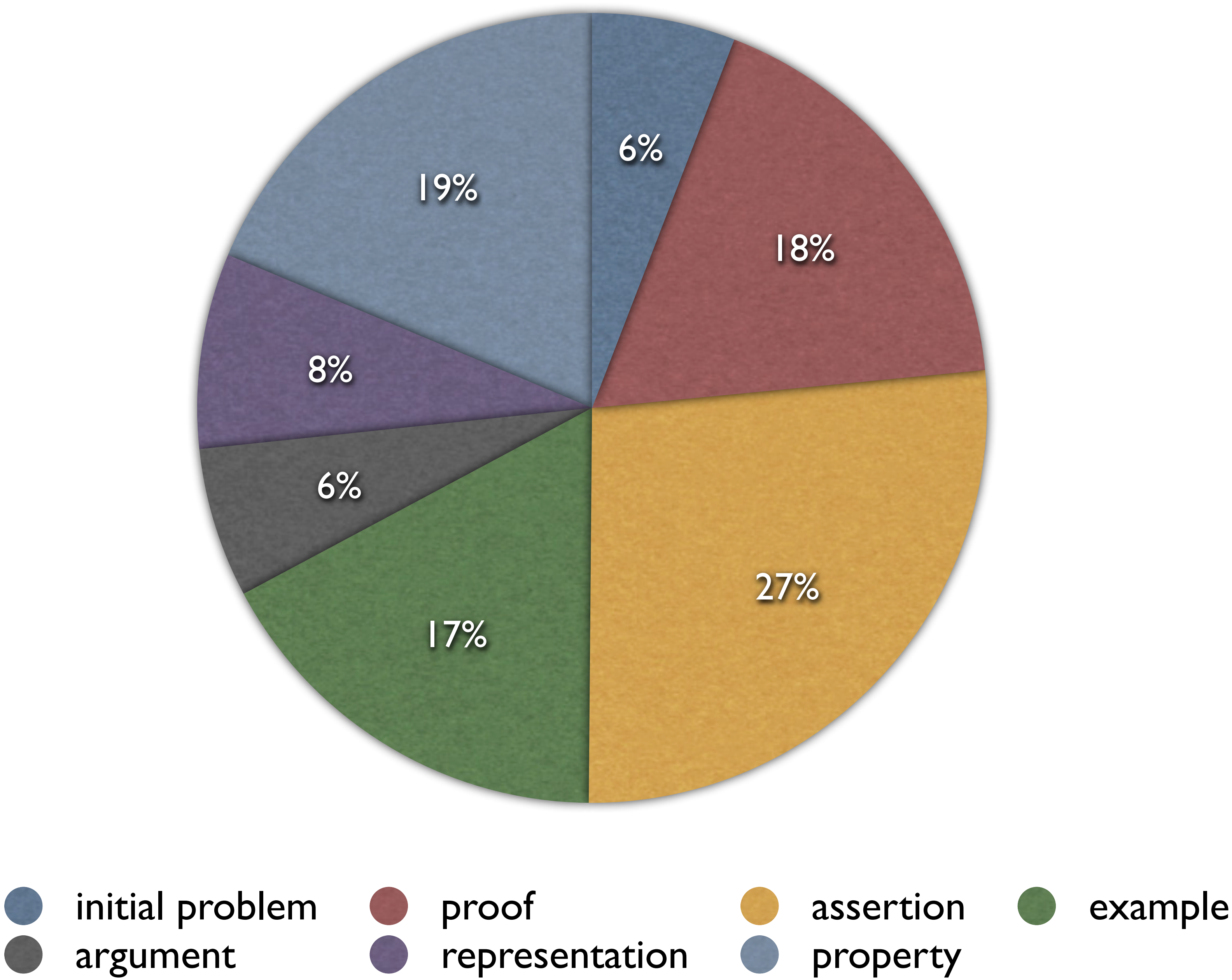
Maths-centred:

1. **Initial problem:** The initial problem is harder if P; The initial problem is hardest when P; Condition C is necessary
2. **Proof:** (approaches) A is not a useful approach; Approaches A and B might be same; Approach A might not work; If we can do X then we have a complete proof; We can't prove X? Proof 1
3. **Assertions:** There is only one of type T; x is not in set S; M is subset of P; Equation E has property P; If we do A then we'll get B; There must always exist X that satisfies condition C2;
4. **Specific cases/instances:** Things get harder in case C; There will always exist instance X that satisfies condition C1; Problem works in instance X; Instance I will be a problem; other cases Y and Z are trivial; case C might be a problem
5. **Arguments:** Let us suppose X. Then Y;
6. **Representation:** there are many ways to write A; by reducing the problem to P;
7. **Property:** This thing has this property, Might not be unique; This thing has this other property; We don't have this property; might have this property; we have this property; This property holds; M has this property; We don't know if it has property P; T must have this other unique property; x doesn't have this property

Total: people category



Total: maths category



	Pa	Pk	Pu	Pv	Mip	Mproof	Mass	Meg	Marg	Mrep	Mprop	Total:
Pa												8%
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Pu												7%
Pv												3%
Mip												8%
Mproof												17%
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Meg												14%
Marg												1%
Mrep												9%
Mprop												11%
Total:	5%	6%	6%	4%	5%	10%	18%	15%	7%	7%	16%	**100**

Ranges for all years 0-4 5-9 10-14 15-19

Part III: hands-on analysis of mathematical and non-mathematical arguments